Twisting Beam Problem

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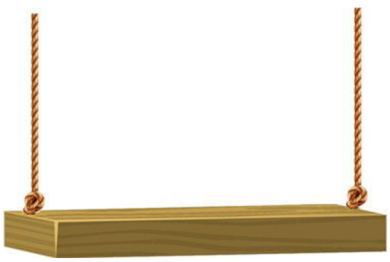

Here is a slightly different kind of problem from the Polish Mathematical Olympiads ([1]).

106. A beam of length a is suspended horizontally by its ends by means of two parallel ropes of lengths b. We twist the beam through an angle φ about the vertical axis passing through the centre of the beam. How far will the beam rise?

My Solution

Figure 1 shows the parmeterization of the problem where h is the height of the beam. Then we get the equation

$$b^2 = \left(\frac{a}{2} - \frac{a}{2}\cos\varphi\right)^2 + \left(-\frac{a}{2}\sin\varphi\right)^2 + (b-h)^2$$

which yields

$$h^2 - 2bh + \frac{a^2}{2}(1-\cos\varphi) = 0$$

From the quadratic formula we obtain

$$h = b - \sqrt{b^2 - \frac{a^2}{2}(1-\cos\varphi)} \qquad (1)$$

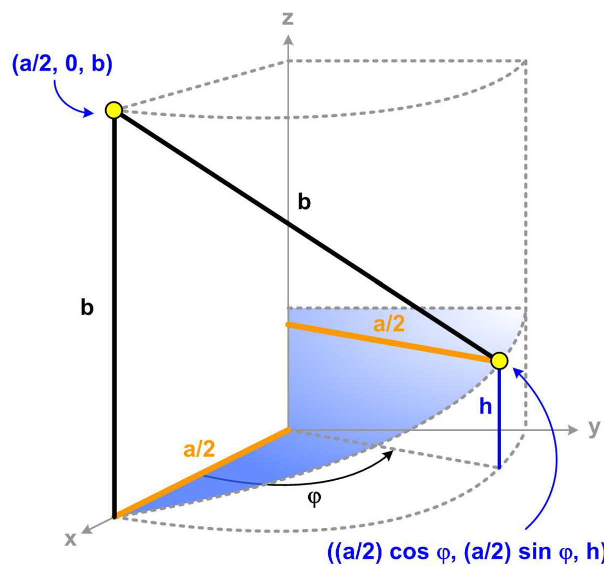

Figure 1

The other solution found by adding the radical to b would mean $h > b$, which is impossible. We only get non-imaginary solutions if

$$b^2 - (a^2/2)(1 - \cos\varphi) \geq 0$$

which happens when

$$\cos\varphi \geq 1 - 2(b/a)^2 \qquad (2)$$

Now when $a \leq b$, for all φ,

$$1 \geq \cos\varphi \geq -1 \geq 1 - 2(b/a)^2,$$

and the minimum $\cos\varphi$ value of -1 occurs when $\varphi = \pi$. Then $b^2 - (a^2/2)(1 - \cos\varphi) = b^2 - a^2$. So from equation (1), for $a \leq b$, h is maximal when twist $\varphi = \pi$ and its value is $h = b - \sqrt{b^2 - a^2}$.

When $a \geq b$,

$$1 \geq \cos\varphi \geq 1 - 2(b/a)^2 \geq -1.$$

As a gets larger, $\cos\varphi$ is squeezed closer to 1 and so the maximal twist φ approaches 0. Choose the maximal value of $\varphi \leq \pi$ where

$$\cos \varphi = 1 - 2(b/a)^2.$$

Then $b^2 - (a^2/2)(1 - \cos \varphi) = 0$ and from equation (1) $h = b$.

Olympiad Solution

Again I will give the Olympiad solution as images of the original. See below p.3. I am not sure I understand what their idea of *oblique* and *orthogonal projections* means as a method of solution. I found the analytic geometry coordinate approach that I used fairly straight-forward and intuitive.

References

[1] Straszewicz, S., *Mathematical Problems and Puzzles from the Polish Mathematical Olympiads*, J. Smolska, tr., Popular Lectures in Mathematics, Vol.12, Pergamon Press, London, 1965 (Polish edition 1960). Problem p.117, solution p.210.

Olympiad Solution

106. In solving geometrical problems a correctly executed drawing is an important aid. Figures in space are shown by means of mappings or projections upon the plane of the drawing. There are various methods of such a mapping. In elementary geometry we usually draw *oblique projections* of figures; in many cases it is convenient to use the *method of orthogonal projections* on two perpendicular planes, i.e. the so called *Monge method*.

We shall present the solution of our problem in two variants, using first one and then the other of the above-mentioned methods of representation.

Method I. Using the method of oblique projection we shall adopt as the plane of projection the plane passing through the beam AB and the suspension points M and N. We draw the quadrilateral $ABNM$ "life size" (Figs. 105 and 106). Let S denote the centre of the beam. When twisted, the beam will assume the position CD. The mid-point of the segment CD lies on the plane of projection; suppose that it is point T.

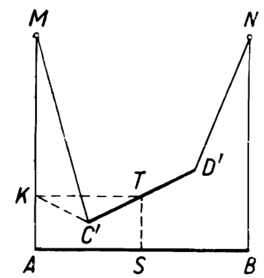

Fig. 105

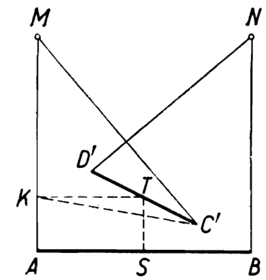

Fig. 106

The position of the projection of point C depends on the direction of projecting; we can regard any point C' as an oblique projection of point C, for example as in Figs. 105 or 106. The projection D' of point D will be a point symmetric to point C' with respect to point T.

The finding of the required length $ST = x$ is simple. We draw a segment TK, parallel and equal to SA; then

$$x = AK = AM - KM.$$

Now $AM = b$, while the segment KM is a side of the right-angled triangle KMC with the hypotenuse $MC = b$ and the other side KC. The segment KC is the base of the isosceles triangle KTC, in which $TK = TC = \frac{1}{2}AB = \frac{1}{2}a$, $\sphericalangle KTC = \varphi$.

Consequently

$$KC = a\sin\frac{\varphi}{2}, \qquad KM = \sqrt{\left(b^2 - a^2\sin^2\frac{\varphi}{2}\right)}.$$

Finally

$$x = b - \sqrt{\left(b^2 - a^2\sin^2\frac{\varphi}{2}\right)}.$$

the same as

$$\boxed{h = b - \sqrt{b^2 - \frac{a^2}{2}(1-\cos\varphi)}}$$

If $b < a$, the torsion angle φ cannot be greater than the angle φ_0 defined by the formula

$$\sin\frac{\varphi_0}{2} = \frac{b}{a} \quad \text{where} \quad \varphi_0 < 180°.$$

For the value $\varphi = \varphi_0$ we have $x = b$. A further enlargement of the angle is not possible without stretching out the ropes.

If $b \geqslant a$, the greatest value of φ is 180°. For $\varphi = 180°$ the ropes cross each other if $b > a$ and coincide if $b = a$.

In the above solution we were concerned with finding the elevation of the beam when twisted. The drawing of the figure in the parallel projection was only a relatively simple illustration necessary for the calculation. If we want the drawing to constitute the graphic solution of the problem, i.e. to give the correct length of the segment ST for given lengths a, b and a given angle φ, we must execute it in a different way. Namely, point T, which in Figs. 105 and 106 was fixed arbitrarily, must be determined by construction from the given magnitudes a, b, φ.

Accordingly, it will be observed that in the right-angled triangle KMC we know the hypotenuse $MC = MA = b$ and the side KC, equal to the base of the isosceles triangle KCT, in which $TK = TC = \frac{1}{2}a$ and $\sphericalangle KTC = \varphi$. From these data we can construct a triangle in order to find the length KM and the length $ST = AM - KM$.

The construction is represented in Fig. 107.

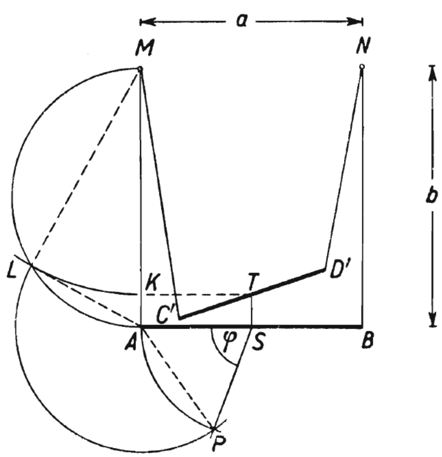

FIG. 107

We construct a triangle ASP in which $AS = SP = \frac{1}{2}a$, $\sphericalangle ASP = \varphi$. We draw a semicircle with diameter AM and a chord $AL = AP$ in it. We mark off on MA a segment MK equal to ML. Point K determines the level of point T, which will be found by drawing a segment $TS = KA$ parallel to AK.

The projection $C'D'$ of the twisted beam will be drawn as before by choosing point C' in an arbitrary manner.

Method II. Figures 108–111 represent the figure in question in the Monge projections for different values of angle φ. The vertical plane of projection is the plane $ABNM$, and the horizontal plane of projection is an arbitrary plane perpendicular to $ABNM$. We shall describe the execution of Fig. 108.

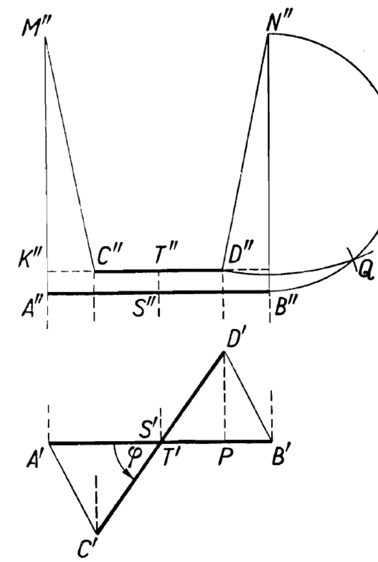

FIG. 108

The projection $C'D'$ is obtained by rotating the segment $A'B'$ about its mid-point S' through the angle φ. The vertical projection of point D lies at such a point D'' of the perpendicular drawn from point D' to $A'B'$ that the length of the segment ND is equal to b, i.e. to the length of $N''B''$.

In order to determine point D'' we consider the right-angled triangle $N''DD''$ formed by the segment $ND = N''D = b$, its vertical projection $N''D''$ and the segment DD'', equal to the distance of point D from the vertical plane of projection, i.e. equal to the distance $D'P$ of the projection D' from $A'B'$. We construct such a triangle taking the segment $N''B'' = b$ as the hypotenuse, describing a circle with diameter $N''B''$ and drawing in it a chord $B''Q = D'P$. Point D'' will lie at the intersection of the circle described from point N'' with radius $N''Q$ and the

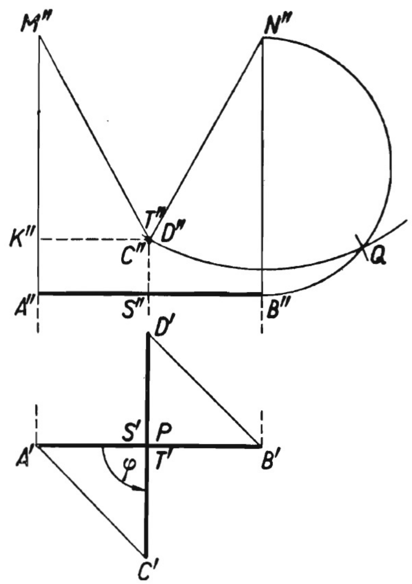

Fig. 109

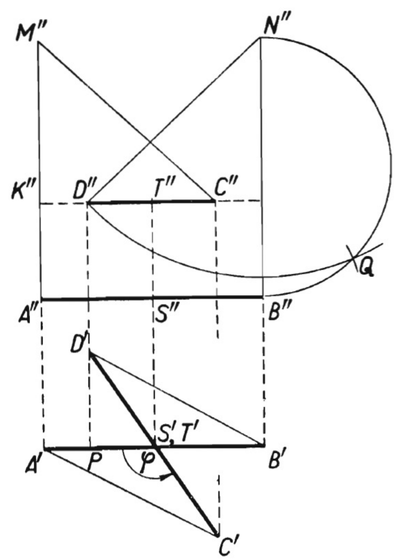

Fig. 110

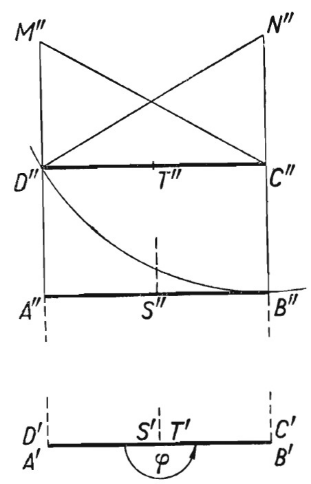

Fig. 111

straight line $D'P$. Point C'' lies symmetrically to point D'' on the other side of the drawing.

Figure 109 for $\varphi = 90°$, Fig. 110 for an obtuse φ, and Fig. 111 for $\varphi = 180°$ are executed in a similar way.

A drawing in the Monge projection executed (on a suitable scale) as shown above gives the graphic solution of our problem, since we obtain in it the required length $x = ST = S''T''$ according to given lengths a, b and a given angle φ.

In a drawing of this kind we can also calculate x. Using Fig. 108 we have

$$x = S''T'' = A''K'' = A''M'' - K''M'' = b - K''M'',$$

$$K''M'' = \sqrt{[(M''C'')^2 - (K''C'')^2]},$$

$$(M''C'')^2 = (N''D'')^2 = (N''Q)^2$$

$$= b^2 - (B''Q)^2 = b^2 - (D'P)^2 = b^2 - \frac{a^2}{4}\sin^2\varphi,$$

$$(K''C'')^2 = (PB')^2 = \left(\frac{a}{2} - \frac{a}{2}\cos\varphi\right)^2 = a^2 \sin^4\frac{\varphi}{2}.$$

Consequently

$$K''M'' = \sqrt{\left(b^2 - \frac{a^2}{4}\sin^2\varphi - a^2\sin^4\frac{\varphi}{2}\right)}$$

$$= \sqrt{\left[b^2 - a^2\sin^2\frac{\varphi}{2}\left(\cos^2\frac{\varphi}{2} + \sin^2\frac{\varphi}{2}\right)\right]}$$

$$= \sqrt{\left(b^2 - a^2\sin^2\frac{\varphi}{2}\right)}.$$

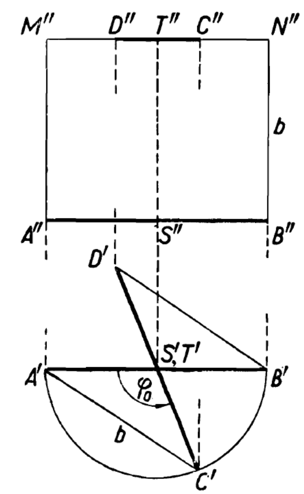

Fig. 112

Finally

$$x = b - \sqrt{\left(b^2 - a^2\sin^2\frac{\varphi}{2}\right)}.$$

Figure 112 corresponds to the case where $b < a$ and the angle $\varphi = \varphi_0$ satisfies the condition $\sin(\varphi_0/2) = b/a$.

In a circle with diameter $A'B'$ we draw chords $A'C' = B'D' = b$. Then $\sphericalangle A'S'C' = \varphi_0$.

The projection $C''D''$ lies on the straight line $M''N''$.