Existence Proofs

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Here is a seemingly simple problem from *Futility Closet* ([1]).

> A quickie from Peter Winkler's *Mathematical Puzzles*, 2021: Can West Virginia be inscribed in a square? That is, is it possible to draw some square each of whose four sides is tangent to this shape?

Technically we might rephrase this as, can we inscribe a flat map of West Virginia in a square, since the boundary of most states is probably not differentiable everywhere, that is, has a tangent everywhere.

But the real significance of the problem is that it is an example of an "existence proof", which in mathematics refers to a proof that asserts the *existence* of a solution to a problem, but does not (or cannot) produce the solution itself. These proofs are second in delight only to the "impossible proofs" which prove that something is impossible, such as trisecting an angle solely with ruler and compass.

Here is another classic example (whose origin I don't recall). Consider the temperatures of the earth around the equator. At any given instant of time there must be at least two antipodal points that have the same temperature. (Antipodal points are the opposite ends of a diameter through the center of the earth.)

Solution

West Virginia map. I'll quote the Futility Closet solution here, since it is the same one I was thinking of.

> Yes. Start by drawing an arbitrary rectangle that hugs the state. Now rotate that rectangle, adjusting its dimensions as it turns to keep each side in contact with the shape. If you turn it through 90 degrees, then the "height" and "width" will have traded places: What had been the shorter side of the rectangle is now the longer. So at some point the two must have had the same value.

The key here is that the circumscribing rectangle is being rotated *continuously*. At first this might not seem possible if the outline of the state is so jagged or in fact a fractal boundary. To see how this might work, consider a rectangle itself and ask if it can be inscribed in a square.

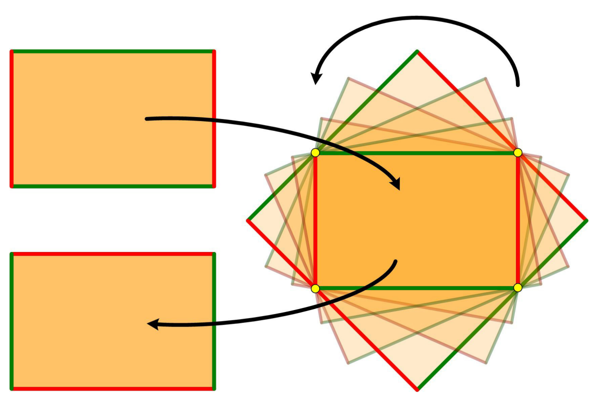


Figure 1

Figure 1 shows a rectangle whose horizontal sides are colored green and vertical sides red. It shows successively rotated rectangles that continue to pass through the corners of the original rectangle. They expand in area until they begin to shrink back to the size of the original rectangle, only now the horizontal sides are red and the vertical sides are green. Notice by symmetry that the maximum rectangle is a square whose diagonals are parallel to the original rectangle. In this case we have actually constructed the desired square. But just by the fact of the continuous motion from the initial to the final rectangle we know that there must have been an intermediate rectangle with sides the same length, that is, a square.[1]

Common antipodal point temperature. (See Figure 2) Let P be an arbitrary point on the equator and let its temperature be T. Let P' be its corresponding antipodal point with temperature T'. Then P' is 180º of longitude from P. Let $D(P) = T - T'$ be the temperature difference. Start at some point P_0 and parmeterize P by its longitudinal distance θ from P_0 so that θ ranges from 0 to 180º while P moves from P_0 to its antipodal point P_0'. If $T_0 = T_0'$, then we are done. Assume $T_0 > T_0'$. (The other case is treated the same.) Then $D(P_0) = D(P(0)) = T_0 - T_0' > 0$. Clearly nearby points have similar temperatures, so we can assume as P moves along the equator that $D(P)$ changes continuously. When P reaches the antipodal point P_0' of the original point P_0, the difference has reversed and is now negative, that is, $D(P_0') = D(P(180º)) = T_0' - T_0 < 0$. Therefore, there must be some point $P^* = P(\theta^*)$ between P_0 and P_0' $(0 < \theta^* < 180º)$ where $D(P^*) = 0$, that is, $T^* = T^{*\prime}$.

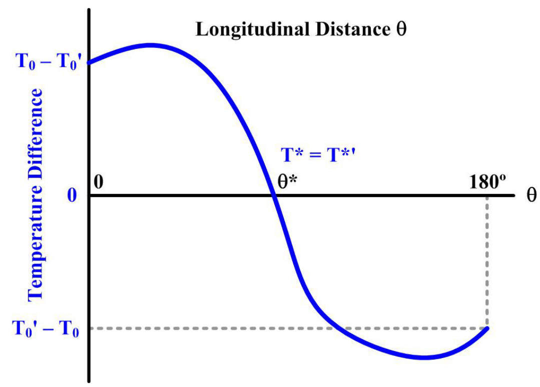


Figure 2

This seems intuitively clear, but it is also the result of the powerful Intermediate Value Theorem for continuous single variable functions in calculus.

Intermediate Value Theorem (IVT)

If $f : [a, b] \to \mathbb{R}$ is a continuous real-valued function on the closed interval $[a, b]$, then f takes on all values between $f(a)$ and $f(b)$, that is, if $f(a) \neq f(b)$, say $f(a) > f(b)$, and if d is such that $f(a) > d > f(b)$, then there is a number c, $a < c < b$, such that $f(c) = d$ (Figure 3).

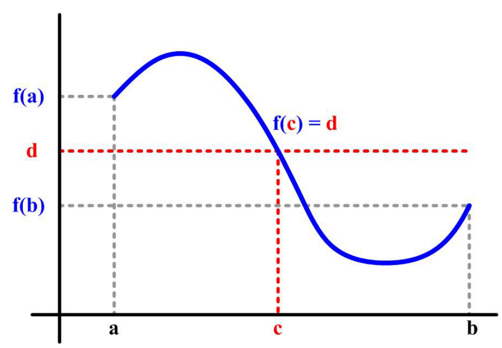


Figure 3

This result can be expanded to say f takes on all values between its maximum M and minimum m on $[a, b]$. (Just find the new values a, b where f equals the max and min, possibly relabeling so that $a < b$.)

It turns out that the IVT is equivalent to the case we treated in the antipodal point problem. That is, if g is continuous on the closed interval $[a, b]$ and $g(a) > 0$ and $g(b) < 0$, then there is a number c, $a < c < b$, such that $g(c) = 0$. (Just take $g(x) = f(x) - d$.)

This is a prime example of an existence proof where we know such a number exists but do not necessarily know what it is.

(A proof of the IVT can be based on the ideas presented in my post on Point Set Topology ([3]). We set up a nested set of closed intervals $[a_k, b_k]$, each half the length of it predecessor, where $f(a_k) > d > f(b_k)$. Just as in the case of the nested intervals that converged on $\sqrt{2}$, these intervals converge on a unique point c in (a, b). So f continuous means $\lim_{k\to\infty} f(a_k) = f(c) \geq d \geq f(c) = \lim_{k\to\infty} f(b_k)$ and so $f(c) = d$.)

1 Burkard Polster at his *Mathologer* Youtube website ([2]) has a nice explicit proof of the existence of the circumscribing square using the ideas we discuss subsequently. His principal interest in the video, however, is to look at a two-dimensional analog of the puzzle, called "the wobbly table problem".

Fixed-Point Theorem (FPT)

A fabulous application of the IVT is a fixed-point theorem for single real variables. Here is a context for such theorems. Suppose you lay an infinitesimally thin string along the edge of a 12-inch ruler. Now pick it up and wad it into a tangle and lay it back down along the ruler, still in an infinitesimally thin line (Figure 4). (That is, keep the situation one-dimensional.) Then some point in the wadded string has to lie over its original position along the ruler. A different wadding produces a different "fixed point". There is no way to wad the string so that all the points on the string move to different locations.

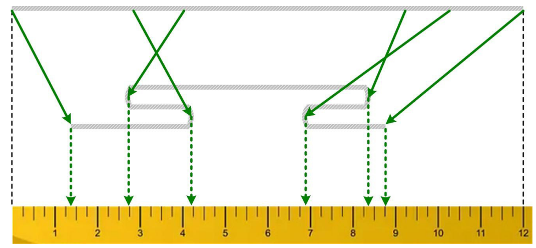


Figure 4 Wadding string along ruler

This is easily shown in Figure 5 (based on the excellent 1966 introductory article by Marvin Shinbrot [4]). If we let x represent the starting positions along the ruler for the string and y the ending positions, then the graph of the function $y = f(x)$ relating the starting positions to the ending positions lies in the 12"x12" square. Since the string is not broken, the function f is continuous. A fixed-point for this function is when $x = f(x)$, or equivalently when the graph of the function $y = f(x)$ intersects the graph of the function $y = x$, which is a straight line from (0, 0) to (12, 12). It is intuitively obvious that we cannot draw a continuous curve from the left edge of the 12"x12" square to the right edge without crossing the diagonal.

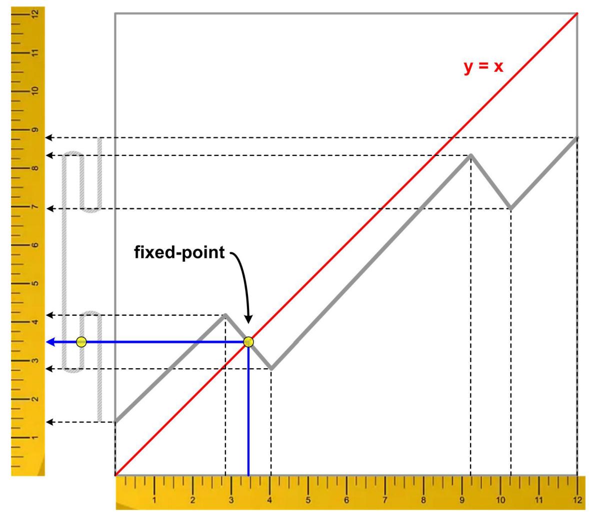


Figure 5 Proving there exists a fixed-point

Analytically, this is just an application of the IVT where $g(x) = f(x) - x$. If $f(0) \neq 0$ and $f(12) \neq 12$, then $g(0) = f(0) - 0 > 0$ and $g(12) = f(12) - 12 < 0$. By the IVT, there is some x^*, $0 < x^* < 12$, such that $g(x^*) = 0$, and so $f(x^*) = x^*$.

So we have a general one-dimensional fixed-point theorem:

Fixed-Point Theorem. If $f : [a, b] \to [a, b]$ is a continuous real-valued function on the closed interval $[a, b]$, then there exists a number c in $[a, b]$ such that $f(c) = c$.

Higher Dimensional Spaces

A two-dimensional analog is Brouwer's fixed-point theorem (B-FPT) ([5]):

Brouwer Fixed-Point Theorem. Every continuous mapping of the closed disk D into itself, $f : D \to D$, has a fixed-point, that is, there is a point p in D such that $f(p) = p$.

A closed disk consists of its circular boundary and all points inside.

A physical example might be the action of stirring a cup of coffee that keeps the surface of the coffee on the surface. At any instant of time some circulated point has to be at its original location. Different instants of time may yield a different fixed point, but there is no way to stir the coffee so that all the points move to a different location. (Apparently this was Brouwer's original inspiration. ([5]))

There are more general results. For example, *any continuous mapping of a closed, bounded, convex, set into itself must have a fixed point*, where "closed" means it contains all its limit points ([3]), "bounded" means all its points are less than a fixed distance from the origin, and "convex" means if two points are in the set, then so are all the points on the line segment joining the points. In fact, this statement holds for spaces of any dimension, even infinite dimensional ones (Banach spaces: Schauder's fixed-point theorem).

Rectangles are also closed, bounded, convex sets and provide additional examples. James Tanton ([6]) illustrates the theorem with the idea of shrinking a map of the US and dropping it inside the larger, original map (Figure 6), even possibly crumpling it in the process. Some point in the smaller map has to lie over its corresponding location on the larger map. Tanton presents one of the standard proofs of the B-FPT using Sperner's Lemma, which has a number of fascinating applications in its own right.

Figure 6

B-FPT Proof. We shall sketch a proof of B-FPT following Shinbrot ([4]) and Brouwer's own, original ideas, though slightly modernized.

We proceed by contradiction. Suppose there is no fixed point. Then for every point P in D, $f(P) \neq P$. Therefore $f(P) - P$ defines a non-zero vector for all P in D. If we attach the vector $f(P) - P$ to each of its corresponding points P, then the distribution of these vectors over D is called a vector field. Since f is continuous, $f(P) - P$ will be continuous, that is, nearby points P, P' are sent to nearby images $f(P)$, $f(P')$, or alternatively the lengths and direction of the vectors vary continuously (Figure 7).

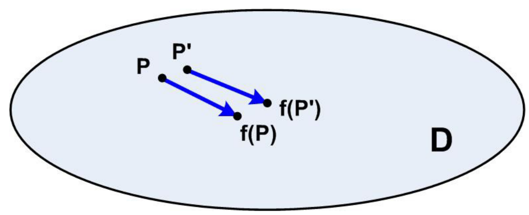


Figure 7

Now consider the behavior of the $f(P) - P$ vector field restricted to the boundary circle of the disk (Figure 8). Each vector attached to the circle has to point inwards, so that as one traverses the circle

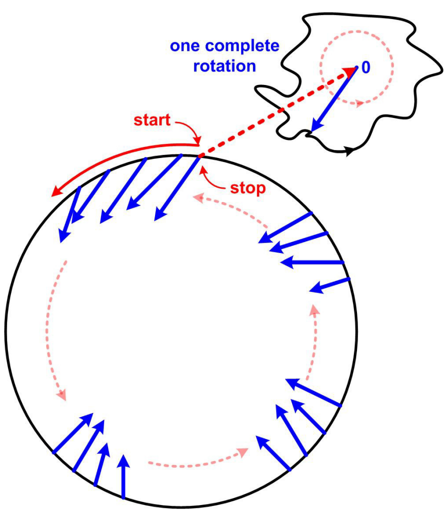


Figure 8

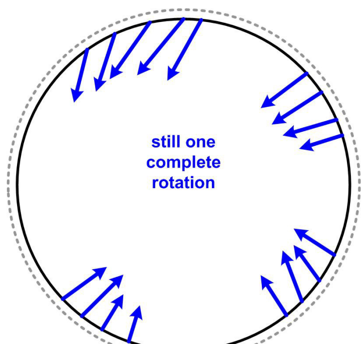


Figure 9

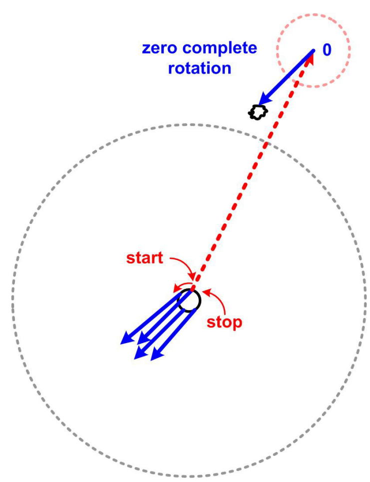


Figure 10

in a counter-clockwise direction, the vectors also rotate counter-clockwise. This is captured in the inset where each vector is translated to a copy whose tails all lie at the origin. Then the heads of the vectors trace out a continuous, closed curve that surrounds the origin, thus making a complete circuit. In this way the corresponding vectors are seen to make one positive (counterclockwise) rotation as the tails of the original vectors traverse the circle in a counterclockwise direction.

Now consider an infinitesimally smaller concentric circle inside the boundary circle of the disk D (Figure 9). By continuity the minimally changed vectors must still make one full rotation, even if they are now not all pointing inside the new circle. As we continue to shrink the concentric circles, the rotation of the vector field along those circles must remain constant, namely, +1.

But when the concentric circle is very small (Figure 10), the nearby vectors should all be pointing in almost the same direction and so make no rotation around the circle, that is, have rotation 0. This contradicts that the rotation should be +1. Therefore, our assumption of no fixed point is false.

Discussion. If we consider again the closed curves defined by the heads of the rotating vectors, we see what is happening. In order to go from the starting curve in Figure 8 to the ending curve in Figure 9, the curves have to continuously shrink without breaking (Figure 11). But at some point they will hit the origin 0. That means some vector must be of zero length (and $f(P) = P$), contrary to our assumption. In fact, there is a theorem that captures this idea ([7] p.114):

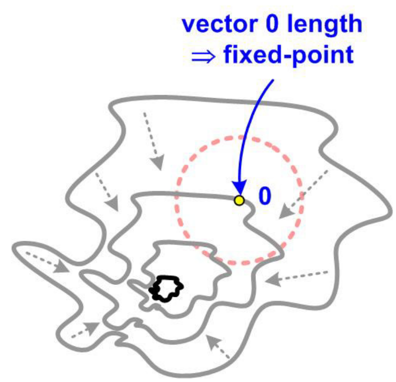


Figure 11

> **Theorem 31.1**. Let V be a continuous vector field defined on a disk D in the plane and such that V_p is not the zero vector for any point p on the boundary circle C of D. If the index of V around C, $I(V, C)$, is not zero, then there is at least one point p in D whose vector V_p is zero.

The index of V around C, $I(V, C)$, is the rotation of the vector field V around C¸ or equivalently the number of windings of the closed curve of tail points of the vector around the origin—called the *winding number*, also known as the *degree of the mapping f*. The continuous shrinking of the curves from one instance to another is called a *homotopy* of the curves and is one of the ideas introduced by Brouwer (along with the degree of a map). It two homotopic closed curves can be shrunk into each other without passing through the origin then they must have the same winding number. In general the winding number can be defined about any point, not just the origin.

Differential Equations

Fixed points and rotations of vector fields have applications to differential equations. Consider a pendulum of length l that swings through an angle θ measured positively from the vertical in the counter-clockwise direction (Figure 12). Then its (second order) differential equation of motion is given by

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0 \qquad (1)$$

If the pendulum is let go at say an angle of $-\pi/4$, it will swing over to $+\pi/4$ under the accelerating force of gravity and then back again, oscillating forever if there is no friction. If the pendulum is let go at $\theta = 0$, it will remain motionless. A slit nudge will barely move it from this position, so it is at a stable equilibrium point. Similarly, if the pendulum is started at $\theta = \pi$, it will also not move—another equilibrium point—but this time a slight nudge will

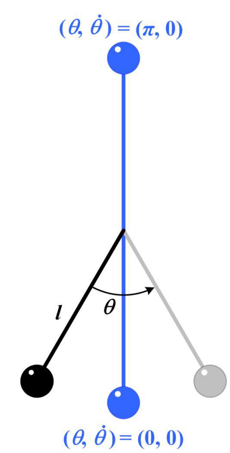


Figure 12 Pendulum

send it swinging around 2π until it comes to rest again. So this is an unstable equilibrium point. Now if the pendulum is started at any angle but with a strong push that sends it spinning around, it will keep spinning (in a frictionless environment).

All these motions can be described in a "phase space diagram" of positions and velocities of the pendulum, that is, coordinates (θ, v) where $v = d\theta/dt$. We reformulate the second order differential equation (1) as a first order vector differential equation. Let $\boldsymbol{\varphi}$ be the vector-valued function of time given by $\boldsymbol{\varphi}(t) = (\theta(t), v(t))$ where $v = d\theta/dt$. Let $\mathbf{F}(\theta, v)$ be a vector field in the phase space given by

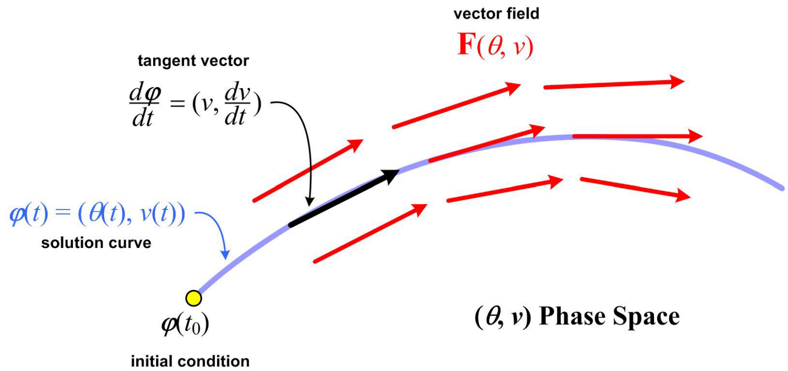


Figure 13 Differential equation $d\varphi / dt = \mathbf{F}(\theta, v)$ in (θ, v) phase space

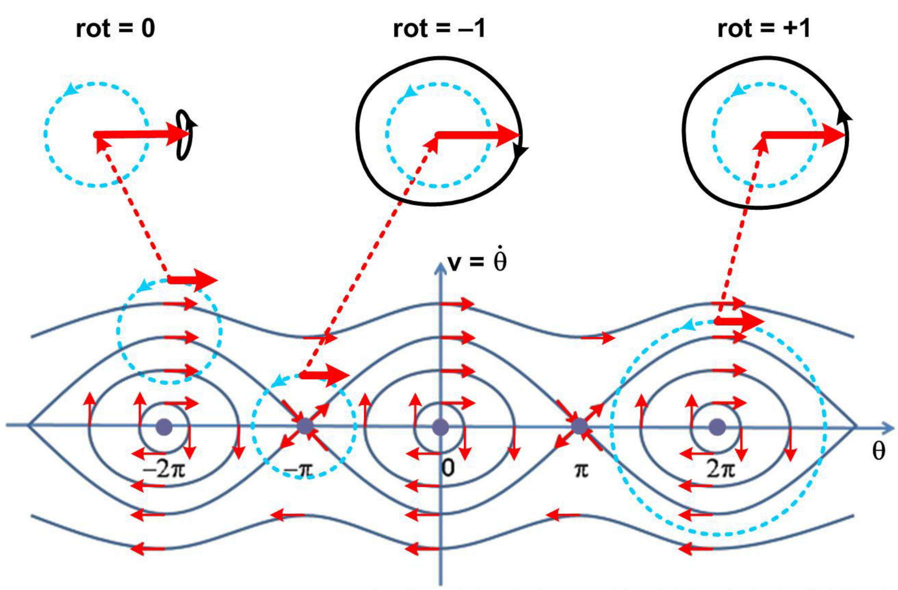
$$\mathbf{F}(\theta, v) = (v, (g/l) \sin \theta).$$

Then the differential equation (1) for the pendulum becomes

$$\frac{d\varphi}{dt} = \begin{pmatrix} d\theta / dt \\ dv / dt \end{pmatrix} = \begin{pmatrix} v \\ (g / l) \sin \theta \end{pmatrix} = \mathbf{F}(\theta, v) \, .$$

This is illustrated geometrically in Figure 13.

The actual phase space for the pendulum is shown in Figure 14 where we also include some local rotations of the vector field $\mathbf{F}$. The large blue dots represent equilibrium points (where both the speed and acceleration of the pendulum are zero), which are zeros of the vector field $\mathbf{F}$, and consequently zeros of the derivative $d\boldsymbol{\varphi}/dt$, called *critical points* of $\boldsymbol{\varphi}$.



http://www-thphys.physics.ox.ac.uk/people/AlexanderSchekochihin/ODE/

Figure 14 (θ, v) Phase space diagram for a pendulum

Notice that in the region of the phase space in which the vector field has zero rotation there are no critical points. But in the regions with non-zero rotations, there are critical points or zeros of the vector field, just as stipulated by **Theorem 31.1** above. The interesting property is that the vanishing points of the vector field with positive rotation are stable equilibria for the pendulum and those with negative rotation are unstable equilibria. Thus the topological properties of the phase space can yield information about the solutions without having to actually solve the equations. This is especially important for nonlinear differential equations, such as the pendulum, for which there are often no easy solutions.

A modern discussion of these ideas, and more, can be found in John Roe's *Winding Around: The Winding Number in Topology, Geometry, and Analysis* (2015) ([8]).

Function Spaces

There is another way of applying fixed point properties to differential equations by looking at corresponding integral equations, as can be seen by the following elementary example. Consider the nonlinear differential equation

$$\frac{dy}{dx} = f(x, y)$$

with initial condition $y_0 = y(x_0)$ where $f(x, y)$ is continuous in some plane region G containing (x_0, y_0). This equation is equivalent to the following integral equation:

$$\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))dt \,.$$

We consider a space C of continuous functions ϕ defined on an appropriate closed subset of G containing (x_0, y_0), and define the mapping $A: C \rightarrow C$ where A is given by

$$A(\phi(x)) = y_0 + \int_{x_0}^{x} f(t, \phi(t))dt \,.$$

Our original differential equation, then, will have a solution when A has a fixed point in C, that is, when $A\phi = \phi$ for some ϕ in C. Thus in a certain sense we have established a correspondence between the analytic properties of differential equations and the topological properties of operators on a space of functions, as we introduced in the Point Set Toplogy posting. An advanced discussion of these ideas can be found in Krasnosel'sky ([9]). But there are a host of references on the subject now that didn't exist when I began studying this subject.

Over the years I have found the ideas in this essay to be some of the most wonderful things I ever encountered in mathematics. The cross fertilization of the geometry of fixed-points and rotations of vector fields with the topology of function spaces and the solutions of nonlinear differential equations is the essence of mathematics. The added spice is the sauciness and impertinence of existence theorems that do not construct the solution.

References

[1] "State House", *Futility Closet*, 15 June 2021 (https://www.futilitycloset.com/2021/06/15/state-house/)

[2] Polster, Burkard, "The Fix-The-Wobbly-Table Theorem", *Mathologer*, 28 July 2018 (https://www.youtube.com/watch?v=aCj3qfQ68m0) Excellent as always. Very clear and simply presented, especially the proof of the "State House" type of problem.

[3] Stevenson, Jim, "Point Set Topology", *Meditations on Mathematics*, 28 December 2018 (https://josmfs.net/2018/12/28/point-set-topology/)

[4] Shinbrot, Marvin, "Fixed-Point Theorems," *Mathematics in the Modern World, Readings from Scientific American*, W. H. Freeman and Co., San Francisco, 1968, pp.145-150 (originally Jan 1966). An early clear explanation in the usual *Scientific American* mode that is unfortunately behind a paywall. I have extracted the essentials. This was my introduction to the fixed-point ideas.

[5] "Brouwer fixed-point theorem" *Wikipedia* (https://en.wikipedia.org/wiki/Brouwer_fixed-point_theorem, 9/2/2021) Comprehensive and informative with interesting historical asides.

[6] Tanton, James, *Freaky Fixed Points*, December 2016 (http://www.jamestanton.com/wp-content/uploads/2016/12/MATH-CIRCLE-SESSION_Freaky-Fixed-Points.pdf) Clear as always. His presentation of Sperner's Lemma is excellent as well. This Lemma is the foundation of a more algebraic topology approach to B-FPT involving simplicial complexes—ideas that are used to extend the results to infinite dimensional spaces.

[7] Chinn, W. G. and N. E. Steenrod, *First Concepts of Topology: The Geometry of Mappings of Segments, Curves, Circles, and Disks*, New Mathematical Library, No.18, Random House, Inc., New York, 1966. Besides fixed-point results, a nice elementary introduction to a number of topics in both point set topology and algebraic topology, and includes classics such as the Ham Sandwich Theorem and the Hairy Ball Theorem.

[8] Roe, John, *Winding Around: The Winding Number in Topology, Geometry, and Analysis*, Student Mathematical Library Vol. 76, American Mathematical Society, 2015. A modern, expansive, more advanced rendition of ideas in this essay.

[9] M. A. Krasnosel'sky, *Topological Methods in the Theory of Nonlinear Integral Equations*, Macmillan, New York, 1964. An early discussion of the degree of a map in infinite dimensional Banach spaces applied to the solutions of nonlinear differential and integral equations. I initially had to read the 1956 version in Russian before the translation was out.