Polynomial Division Problem

11 August 2021

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Here is a challenging problem from the Polish Mathematical Olympiads published in 1960 ([1])

22. Prove that the polynomial

$$x^{44} + x^{33} + x^{22} + x^{11} + 1$$

is divisible by the polynomial $x^4 + x^3 + x^2 + x + 1$.

My Solution

Trying to divide the polynomials directly would be an enormous headache, so mathematicians being lazy, I sought a different route. One idea would be to somehow use the fact that if $x = \alpha$ is a root of a polynomial equation $p(x) = 0$, then $(x - \alpha)$ is a factor of $p(x)$.

Let $q(x) = x^{44} + x^{33} + x^{22} + x^{11} + 1$ and $p(x) = x^4 + x^3 + x^2 + x + 1$. The second polynomial suggests an nth root of unity approach since

$$(x - 1)\, p(x) = (x - 1)\, (x^4 + x^3 + x^2 + x + 1) = x^5 - 1.$$

The roots of $(x - 1)\, p(x) = 0$ are therefore all the 5^{th} roots of unity, namely,

$$x_k = e^{i2\pi k/5} \text{ for } k = 0, 1, 2, 3, 4.$$

Then the four roots of $p(x) = 0$ are x_k for $k = 1, 2, 3, 4$, and $p(x)$ factors into

$$p(x) = (x - x_1)\, (x - x_2)\, (x - x_3)\, (x - x_4).$$

If we can show for $k = 1, 2, 3, 4$, the x_k are roots of $q(x) = 0$, then $p(x)$ divides $q(x)$. Now

$$0 = q(x) = x^{44} + x^{33} + x^{22} + x^{11} + 1 = (x^{11})^4 + (x^{11})^3 + (x^{11})^2 + (x^{11}) + 1$$

and $$x_k^{11} = x_k^5\, x_k^5\, x_k = 1 \cdot 1 \cdot x_k \qquad \text{for } k = 1, 2, 3, 4.$$

So $$q(x_k) = p(x_k) = 0 \qquad \text{for } k = 1, 2, 3, 4,$$

and we are done.

Olympiads Solution

Given the voluminous computations involved, I reproduce images of their solution rather than retype it. See below p.2.

References

[1] Straszewicz, S., *Mathematical Problems and Puzzles from the Polish Mathematical Olympiads*, J. Smolska, tr., Popular Lectures in Mathematics, Vol.12, Pergamon Press, London, 1965 (Polish edition 1960). Problem p.4, solution p.29.

Olympiads Solution

22. Prove that the polynomial

$$x^{44}+x^{33}+x^{22}+x^{11}+1$$

is divisible by the polynomial $x^4+x^3+x^2+x+1$.

22. *Method I.* We shall assume the well-known theorem of algebra which states that the binomial a^n-1 (where n is a natural number) is divisible by the binomial $a-1$, i.e. that

$$a^n-1 = (a-1)(a^{n-1}+a^{n-2}+\ldots+1). \qquad (1)$$

The second factor on the right-hand side of formula (1) is a geometrical progression, of which only the first two terms and the last term are written down, the remaining terms being replaced by dots.

The solution of the problem can be obtained by applying formula (1) to the binomial $x^{55}-1$ in two ways. First, if we substitute $a = x^{11}$, $n = 5$ in formula (1), we obtain the equality

$$x^{55}-1 = (x^{11}-1)(x^{44}+x^{33}+x^{22}+x^{11}+1),$$

and applying formula (1) to the factor $x^{11}-1$ we have

$$x^{55}-1 = (x-1)(x^{10}+x^9+\ \ldots\ +1)(x^{44}+x^{33}+\ldots+1). \qquad (2)$$

If, on the other hand, we substitute $a = x^5$, $n = 11$ in formula (1), we obtain the equality

$$x^{55}-1 = (x^5-1)(x^{50}+x^{45}+\ldots+1),$$

which, on the application of formula (1) to the factor x^5-1, gives the equality

$$x^{55}-1 = (x-1)(x^4+x^3+x^2+x+1)(x^{50}+x^{45}+\ldots+1). \qquad (3)$$

Equalities (2) and (3) imply

$$(x^{10}+x^9+\ldots+1)(x^{44}+x^{33}+\ldots+1)$$
$$= (x^4+x^3+\ldots+1)(x^{50}+x^{45}+\ldots+1). \qquad (4)$$

From this equality we shall derive the required theorem; we first transform the first factor of the left-hand side:

$$x^{10}+x^9+\ldots+1$$
$$= (x^{10}+x^9+\ldots+x^6)+(x^5+x^4+\ldots+x)+1$$
$$= x^6(x^4+x^3+\ldots+1)+x(x^4+x^3+\ldots+1)+1$$
$$= (x^4+x^3+\ldots+1)(x^6+x)+1.$$

Formula (4) can therefore be written in the form

$$(x^4+x^3+\ldots+1)(x^6+x)(x^{44}+x^{33}+\ldots+1)+(x^{44}+x^{33}+\ldots+1)$$
$$= (x^4+x^3+\ldots+1)(x^{50}+x^{45}+\ldots+1).$$

Let us subtract the first term of the equality from each side, and then let us factorize the right-hand side; we obtain

$$x^{44}+x^{33}+\ldots+1 = (x^4+x^3+\ldots+1)[x^{50}+x^{45}+\ldots+1-$$
$$-(x^6+x)(x^{44}+x^{33}+\ldots+1)]. \qquad (5)$$

Equality (5) shows that the polynomial $x^{44}+x^{33}+x^{22}+x^{11}+1$ is divisible by the polynomial $x^4+x^3+x^2+x+1$.

The polynomial appearing in the square brackets on the right-hand side of formula (5) could be ordered, on opening the round brackets, according to the powers of x; this, however, is not necessary for the proof of the theorem. We shall only remark that the terms containing x^{50} and x^{45} are reduced and consequently formula (5) can be written in a simpler form:

$$x^{44}+x^{33}+\ldots+1 = (x^4+x^3+\ldots+1)[x^{40}+{}^{35}+\ldots+1-$$
$$-(x^6+x)(x^{33}+x^{22}+x^{11}+1)]. \qquad (6)$$

Method II. Let us multiply each of the given polynomials

$$f(x) = x^{44}+x^{33}+x^{22}+x^{11}+1$$

and

$$g(x) = x^4+x^3+x^2+x+1$$

by $x-1$; we shall obtain the polynomials

$$F(x) = (x-1)(x^{44}+x^{33}+x^{22}+x^{11}+1)$$
$$= x^{45}-x^{44}+x^{34}-x^{33}+x^{23}-x^{22}+x^{12}-x^{11}+x-1$$

and

$$G(x) = (x-1)(x^4+x^3+x^2+x+1) = x^5-1.$$

In order to prove that the polynomial $f(x)$ is divisible by the polynomial $g(x)$ it is sufficient to show that the polynomial $F(x)$ is divisible by the polynomial $G(x)$, i.e. by x^5-1.

Now

$$x^{45}-x^{44}+x^{34}-x^{33}+x^{23}-x^{22}+x^{12}-x^{11}+x-1$$
$$= (x^{45}-1)-x^{34}(x^{10}-1)-x^{23}(x^{10}-1)-x^{12})(x^{10}-1)-$$
$$-x(x^{10}-1) = [(x^5)^9-1]-(x^{10}-1)(x^{34}+x^{23}+x^{12}+x)$$
$$= (x^5-1)(x^{40}+x^{35}+\ldots+1)-(x^5-1)(x^5+1)(x^{34}+x^{23}+x^{12}+x)$$
$$= (x^5-1)\,[x^{40}+x^{35}+\ldots+1-(x^5+1)(x^{34}+x^{23}+x^{12}+x)].$$

Consequently

$$F(x) = G(x)\,[x^{40}+x^{35}+\ldots+1-(x^6+x)(x^{33}+x^{22}+x^{11}+1)].$$

Thus the polynomial $F(x)$ is indeed divisible by the polynomial $G(x)$, which is what we wanted to prove.

Dividing both sides of the last equality by $x-1$, we obtain the formula

$$x^{44}+x^{33}+x^{22}+x^{11}+1 = (x^4+x^3+x^2+x+1)[x^{40}+x^{35}+\ldots+$$
$$+1-(x^6+x)(x^{33}+x^{22}+x^{11}+1)],$$

i.e. the formula (6) obtained by method I.

Method III. A very short and simple solution of the problem will be obtained by the use of complex numbers and their geometrical representation on a plane. If x denotes a complex variable, then the roots of the equation

$$x^n-1 = 0$$

can be represented as the vertices of a regular n-gon $A_1A_2\ldots A_n$ inscribed in a unit circle C, drawn in the plane of complex numbers x from the origin 0, the vertex A_n of the polygon lying at point $x = 1$.

Let us write, as in method I:

$$x^{55}-1 = (x^{11}-1)(x^{44}+x^{33}+x^{22}+x^{11}+1),$$
$$x^5-1 = (x-1)(x^4+x^3+x^2+x+1).$$

The roots of the binomial $x^{55}-1$ correspond to the vertices A_1, $A_2, \ldots, A_{55}$ of a regular 55-gon inscribed in the above-mentioned circle C, the root $x = 1$ corresponding to the vertex A_{55} (Fig. 3). Similarly, the roots of the binomial $x^{11}-1$ correspond to the vertices of a regular 11-gon; those vertices are to be found among the vertices of that 55-gon; namely they are the points $A_5, A_{10}, A_{15}, \ldots, A_{55}$. The roots of the polynomial $x^{44}+x^{33}+$ $+x^{22}+x^{11}+1$ correspond to those vertices of the 55-gon which will remain after we have rejected the vertices of the 11-gon; there are 44 of them and the vertices $A_{11}, A_{22}, A_{33}, A_{44}$ are among them.

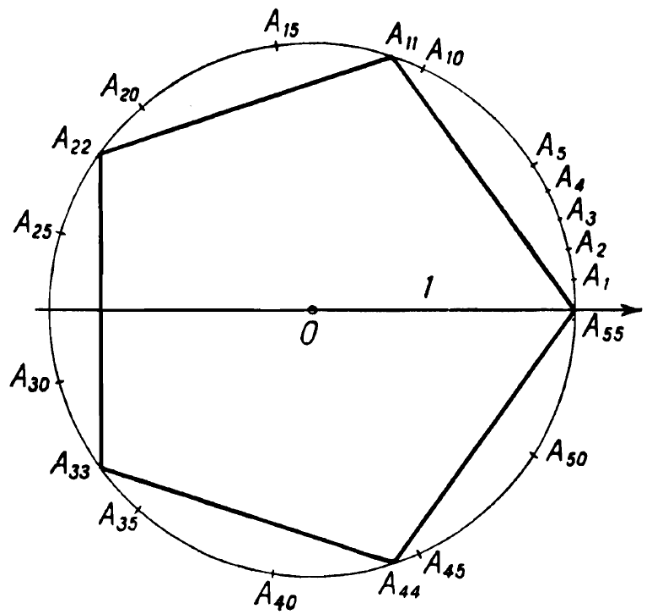

FIG. 3

On the other hand, to the roots of the binomial x^5-1 correspond the vertices of a regular 5-gon, which are also to be found among the vertices of the 55-gon: they are the points A_{11}, A_{22}, A_{33}, A_{44}, A_{55}; to the roots of the polynomial $x^4+x^3+x^2+x+1$ correspond only the vertices A_{11}, A_{22}, A_{33}, A_{44}, because we must reject the vertex A_{55}, corresponding to the number $x = 1$.

Apparently the roots of the polynomial $x^4+x^3+x^2+x+1$ are at the same time roots of the polynomial $x^{44}+x^{33}+x^{22}+x^{11}+1$, which implies that the latter polynomial is divisible by the former.