Bottema's Theorem

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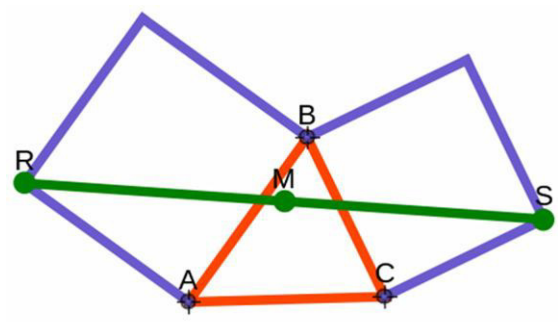

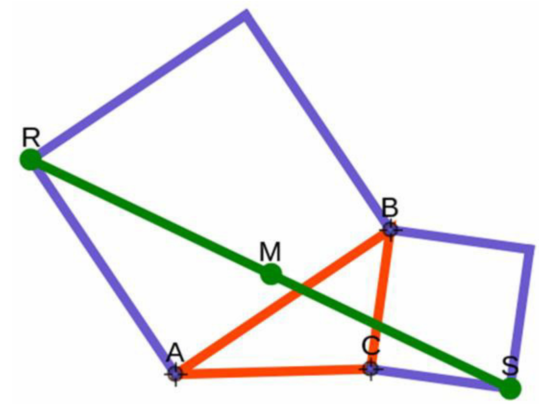

This seemingly magical result from *Futility Closet* ([1]) defies proof at first. Go to the Wolfram demo by Jay Warendorff[1] and then …

Grab point B above and drag it to a new location. Surprisingly, M, the midpoint of RS, doesn't move.

This works for any triangle — draw squares on two of its sides, note their common vertex, and draw a line that connects the vertices of the respective squares that lie opposite that point. Now changing the location of the common vertex does not change the location of the midpoint of the line.

It was discovered by Dutch mathematician Oene Bottema.

As we shall see, Bottema's Theorem has shown up in other guises as well.

Solution

This is one of the few times I was able to capture the train of thought and associations that led me to the solution. First, I realized the critical elements of the figure are the lower halves of the two squares that are pivoting around the fixed points A and C. I began wondering if parameterizing the figure with complex numbers would help. And then the association hit me. I had seen this problem before—a buried treasure problem in Gamow's 1947 book, *One, Two, Three … Infinity* ([2], it was so much fun that I am including it in an Appendix below, p.3). I recalled it did use complex variables and I tried to rederive the solution for myself.

Without loss of generality we place vertex A at the coordinate (–1, 0), vertex C at (1, 0) along the horizontal (real) axis, and vertex B at an arbitrary point (a, b) (Figure 1). We can also express these points as complex numbers: A = –1, C = 1, and B = $a + i\, b$. Since complex numbers add like vectors componentwise ($(a + i\, b) + (c + i\, d) = (a + c) + i\, (b + d)$), we can also think of A, B, and C as vectors from the origin (0, 0). But there is also complex multiplication that has an effect on complex numbers that is different from multiplying a vector by a real number. In particular, multiplying a complex number by i rotates its corresponding vector 90º counter-clockwise.[2] Three successive multiplications by $i = -i$ rotates the corresponding vector 270º counter-clockwise or –90º clockwise.

[1] https://demonstrations.wolfram.com/BottemasTheorem/#embed

[2] See my post "Complex Numbers – Geometric Viewpoint" (http://josmfs.net/2019/01/07/complex-numbers-geometric-viewpoint/)

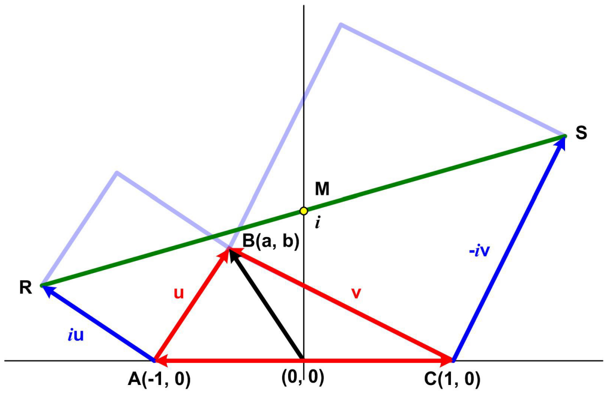

Figure 1

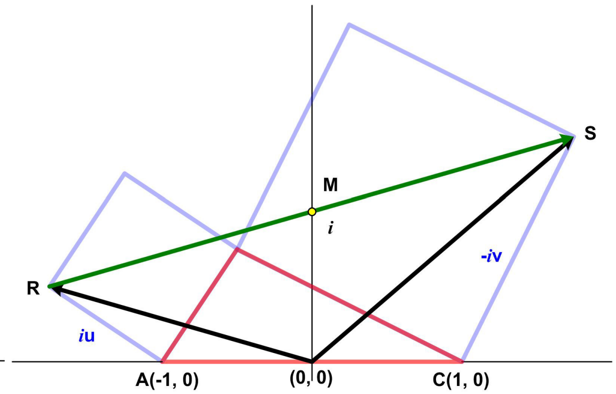

Figure 2

Let **u** = B – A = B + 1 and **v** = B – C = B – 1 (Figure 1). Then rotating the vectors by the appropriate ±90º yields *i* **u** = *i* B + *i* and –*i* **v** = –*i* B + *i* . Therefore, we can write R and S as complex numbers (vectors from the origin) as (Figure 2)

$$R = A + i\,\mathbf{u} = -1 + i\,B + i$$

$$S = C + (-i\,\mathbf{v}) = +1 - i\,B + i$$

Now the midpoint M of the line between R and S is given vectorially by

$$M = R + (S - R)/2 = (R + S)/2 = (2i)/2 = i$$

And so the midpoint M is independent of the location of the point B.

History. The South African educator, Michael de Villiers, noted some history of Bottema's Theorem, which he said was published in 1938 ([3]):

As is often the case in mathematics, though given Bottema's name, the result was also known elsewhere at about the same time, and already known and mentioned earlier. It is an exercise in C V Durell's *New Geometry for Schools* (1939), p. 287, Q26. Earlier still, it is in J W Russell's *Sequel to Elementary Geometry* (1907), p. 34, Sect 6 — a worked example. The converse result is an exercise in a French textbook, *Traite de Geometrie* by E Rouche and C de Comberousse (1900) vol 1, p. 395, Q254. This is the 7th edition, and the result might have appeared even earlier.

More information can be found at *Wikipedia* ([4]), including the reference to Atara Shriki's "Back to Treasure Island" ([5]), which references Gamow's solution and has some purely geometric solutions as well.

References

[1] "Bottema's Theorem", *Futility Closet*, 22 June 2021 (https://www.futilitycloset.com/2021/06/22/bottemas-theorem/)

[2] Gamow, George, *One Two Three … Infinity: Facts and Speculations of Science*, Dover Publications Ind., New York, 1947, rev 1961, 1988 pp.35-38

[3] de Villiers, Michael, "Bottema's Theorem" (http://dynamicmathematicslearning.com/bottema.html)

[4] "Bottema's theorem", *Wikipedia* (https://en.wikipedia.org/wiki/Bottema%27s_theorem)

[5] Shriki, A., "Back to Treasure Island", *The Mathematics Teacher,* **104** (9): 658–664, JSTOR 20876991, 2011 (https://www.researchgate.net/publication/259860311_Back_to_TREASURE_ISLAND)

Appendix

Buried Treasure

If you still feel a veil of mystery surrounding imaginary numbers you will probably be able to disperse it by working out a simple problem in which they have practical application.

There was a young and adventurous man who found among his great-grandfather's papers a piece of parchment that revealed the location of a hidden treasure. The instructions read:

> *"Sail to _______ North latitude and ___________ West longitude[3] where thou wilt find a deserted island. There lieth a large meadow, not pent, on the north shore of the island where standeth a lonely oak and a lonely pine.[4] There thou wilt see also an old gallows on which we once were wont to hang traitors. Start thou from the gallows and walk to the oak counting thy steps. At the oak thou must turn right by a right angle and take the same number of steps. Put here a spike in the ground. Now must thou return to the gallows and walk to the pine counting thy steps. At the pine thou must turn left by a right angle and see that thou takest the same number of steps, and put another spike into the ground. Dig halfway between the spikes; the treasure is there."*

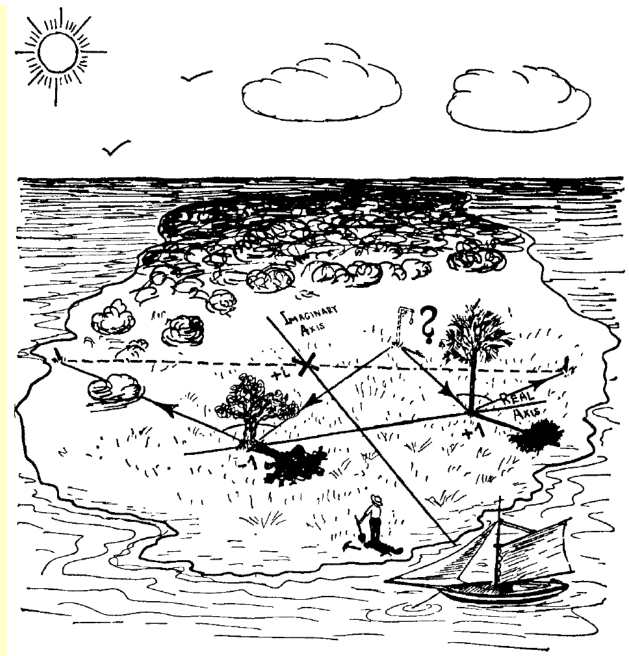

FIGURE 11 Treasure hunt with imaginary numbers.

The instructions were quite clear and explicit, so our young man chartered a ship and sailed to the South Seas. He found the island, the field, the oak and the pine, but to his great sorrow the gallows was gone. Too long a time had passed since the document had been written; rain and sun and wind had disintegrated the wood and returned it to the soil, leaving no trace even of the place where it once had stood.

Our adventurous young man fell into despair, then in an angry frenzy began to dig at random all over the field. But all his efforts were in vain; the island was too big! So he sailed back with empty hands. And the treasure is probably still there.

A sad story, but what is sadder still is the fact that the fellow might have had the treasure, if only he had known a bit about mathematics, and specifically the use of imaginary numbers. Let us see if we can find the treasure for him, even though it is too late to do him any good.

Consider the island as a plane of complex numbers; draw one axis (the real one) through the base of the two trees, and another axis (the imaginary one) at right angles to the first, through a point half

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[3] The actual figures of longitude and latitude were given in the document but are omitted in this text, in order not to give away the secret.

[4] The names of the trees are also changed for the same reason as above. Obviously there would be other varieties of trees on a tropical treasure island.

way between the trees (Figure 11). Taking one half of the distance between the trees as our unit of length, we can say that the oak is located at the point –1 on the real axis, and the pine at the point +1. We do not know where the gallows was so let us denote its hypothetical location by the Greek letter Γ (capital gamma), which even looks like a gallows. Since the gallows was not necessarily on one of the two axes Γ must be considered as a complex number: $\Gamma = a + bi$, in which the meaning of a and b is explained by Figure 11.

Now let us do some simple calculations remembering the rules of imaginary multiplication as stated above.[5] If the gallows is at Γ and the oak at –1, their separation in distance and direction may be denoted by $(-1) - \Gamma = -(1 + \Gamma)$. Similarly the separation of the gallows and the pine is $1 - \Gamma$. To turn these two distances by right angles clockwise (to the right) and counterclockwise (to the left) we must, according to the above rules multiply them by $-i$ and by i, thus finding the location at which we must place our two spikes as follows:[6]

$$\text{first spike: } (-i)[-(1+\Gamma)] + (-1) = i(\Gamma + 1) - 1$$

$$\text{second spike: } (+i)(1 - \Gamma) + 1 = i(1 - \Gamma) + 1$$

Since the treasure is halfway between the spikes, we must now find one half the sum of the two above complex numbers. We get:

$$\tfrac{1}{2}\,[i(\Gamma + 1) - 1 + i(1 - \Gamma) + 1] = \tfrac{1}{2}\,[+i\Gamma + i - 1 + i - i\Gamma + 1] = \tfrac{1}{2}\,(+2i) = +i.$$ [7]

We now see that the unknown position of the gallows denoted by Γ fell out of our calculations somewhere along the way, and that, regardless of where the gallows stood, the treasure must be located at the point $+i$.

And so, if our adventurous young man could have done this simple bit of mathematics, he would not have needed to dig up the entire island, but would have looked for the treasure at the point indicated by the cross in Figure 11, and there would have found the treasure.

If you still do not believe that it is absolutely unnecessary to know the position of the gallows in order to find the treasure, mark on a sheet of paper the positions of two trees, and try to carry out the instructions given in the message on the parchment by assuming several different positions for the gallows. You will always get the same point, corresponding to the number $+i$ on the complex plane!

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[5] JOS: $(a + ib) + (c + id) = (a + c) + i(b + d)$, $i^2 = -1$, and $(a + ib)(c + id) = (ac - bd) + i(ad + bc)$

[6] JOS: I corrected the erroneous signs. The original was:

$$\text{first spike: } (-i)[-(1+\Gamma)] + 1 = i(\Gamma + 1) - 1$$

$$\text{second spike: } (+i)(1 - \Gamma) - 1 = i(1 - \Gamma) + 1$$

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$$\tfrac{1}{2}\,[i(\Gamma + 1) + 1 + i(1 - \Gamma) - 1] = \tfrac{1}{2}\,[+i\Gamma + i + 1 + i - i\Gamma - 1] = \tfrac{1}{2}\,(+2i) = +i$$