Area vs. Perimeter Puzzle

5 June 2021

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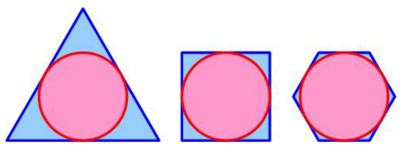

This surprising, but simple, puzzle is from the 12 April MathsMonday offering[1] by MEI, an independent curriculum development body for mathematics education in the UK.

In the diagram various regular polygons, P, have been drawn whose sides are tangents to a circle, C. Show that for any regular polygon drawn in this way:

$$\frac{\text{Area of P}}{\text{Perimeter of P}} = \frac{\text{Area of C}}{\text{Circumference of C}}$$

Given that the polygons approximate the circle in the limit, it would not be surprising that this relationship would hold—*in the limit*. It *is* surprising that it should be true for every regular polygon that circumscribes the circle.

Solution

Let P be a regular polygon of n sides circumscribing a circle C of radius r. Let A be the area of the polygon, L the length of the perimeter of the polygon, and s the length of a side. Then $L = ns$. Consider the triangle of area T shown in Figure 1. Then

$$T = ½\ rs = ½\ rL/n$$

and

$$A = n(½\ rL/n) = ½\ rL$$

So

$$A/L = ½\ r$$

This is a constant independent of the number of sides of the polygon.

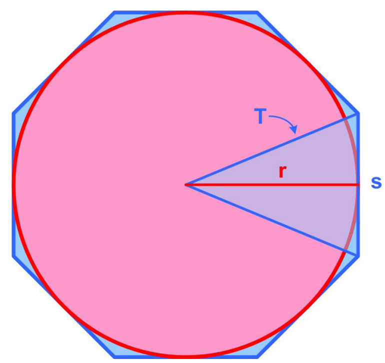

Figure 1

Now the ratio of the area of the circle to its circumference is

$$\pi r^2 / 2\pi r = r/2$$

and so we have the desired equivalence.

Rather amazing. Actually, the surprising thing is why I had never come across this simple fact before. Maybe it is common knowledge and I just missed it.

[1] https://twitter.com/Beamathsteacher/status/1381577925657571330