Wine Into Water Problem

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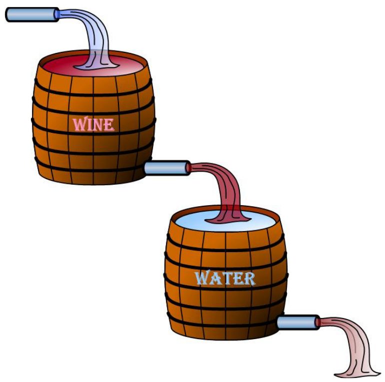
Jim Stevenson



Here is a challenging problem from the 1874 *The Analyst* ([2]).

> 4. A cask containing a gallons of wine stands on another containing a gallons of water; they are connected by a pipe through which, when open, the wine can escape into the lower cask at the rate of c gallons per minute, and through a pipe in the lower cask the mixture can escape at the same rate; also, water can be let in through a pipe on the top of the upper cask at a like rate. If all the pipes be opened at the same instant, how much wine will be in the lower cask at the end of t minutes, supposing the fluids to mingle perfectly?
>
> — Communicated by Artemas Martin, Mathematical Editor of *Schoolday* Magazine, Erie, Pennsylvania.

I found the problem in Benjamin Wardhaugh's book ([1]) where he describes *The Analyst*:

> Beginning in 1874 and continuing as *Annals of Mathematics* from 1884 onward, *The Analyst* appeared monthly, published in Des Moines, Iowa, and was intended as "a suitable medium of communication between a large class of investigators and students in science, comprising the various grades from the students in our high schools and colleges to the college professor." It carried a range of mathematical articles, both pure and applied, and a regular series of mathematical problems of varying difficulty: on the whole they seem harder than those in *The Ladies' Diary* and possibly easier than the *Mathematical Challenges* in the extract after the next. Those given here appeared in the very first issue.

I tailored my solution after the "Diluted Wine Puzzle",[1] though this problem was more complicated. Moreover, the final solution must pass from discreet steps to continuous ones.

There is a bonus problem in a later issue ([4]):

> 19. Referring to Question 4, (No. 1): At what time will the lower cask contain the greatest quantity of wine? —Communicated by Prof. Geo. R. Perkins.

My Solution

In order to know what happens instantaneously at time t, we will first reduce the problem to discrete steps. That is, we will subdivide time t into n discrete intervals of length Δt. To obtain the continuous case, we will take the limit as $\Delta t \to 0$ and $n = t/\Delta t \to \infty$.

Table 1 shows what happens in each discrete step i. V_i is the amount of wine (vino) and W_i the amount of water in cask 1 at the end of the ith interval. V'_i is the amount of wine and W'_i the amount of water in cask 2 at the end of the ith interval. What we are ultimately interested in is the behavior of the wine in cask 2, V'_i.

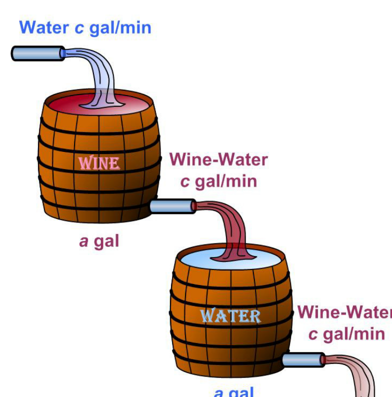


Figure 1 Problem Statement

[1] http://josmfs.net/2019/01/23/diluted-wine-puzzle/

Table 1 Fluid Flow Stages

Time (min)	Water In (gals)	Cask 1			Liquid Out/In (gals)	Cask 2			Liquid Out (gals)
		Wine	**Water**	**Vol** $\mathbf{V_i+W_i}$		**Wine**	**Water**	**Vol** $\mathbf{V'_i+W'_i}$	
0		$V_0 = a$	$W_0 = 0$	a		$V'_0 = 0$	$W'_0 = a$	a	
Δt	$c\ \Delta t$ →	↓ $\underline{V}_1 = V_0$ ↓ $V_1 = (1-r)V_0$ $\boxed{r = {}^{c\Delta t}/_{a+c\Delta t}}$	↓ $\underline{W}_1 = W_0 + c\Delta t = c\Delta t$ ↓ $W_1 = (1-r)(W_0 + c\Delta t)$	$a+c\Delta t$ a	$c\ \Delta t =$ $r\ \underline{V}_1 + r\ \underline{W}_1$ →	↓ $\underline{V'}_1 = V'_0 + r\underline{V}_1$ $= V'_0 + rV_0$ ↓ $V'_1 = (1-r)(V'_0 + rV_0)$	↓ $\underline{W'}_1 = W'_0 + r\underline{W}_1$ $= W'_0 + r(W_0 + c\Delta t)$ ↓ $W'_1 = (1-r)(W'_0 + r(W_0 + c\Delta t))$	$a+c\Delta t$ a	$c\ \Delta t =$ $r\ \underline{V'}_1 + r\ \underline{W'}_1$ →
$2\ \Delta t$	$c\ \Delta t$ →	↓ $\underline{V}_2 = V_1$ ↓ $V_2 = (1-r)V_1$	↓ $\underline{W}_2 = W_1 + c\Delta t$ ↓ $W_2 = (1-r)\ (W_1 + c\Delta t)$	$a+c\Delta t$ a	$c\ \Delta t =$ $r\ \underline{V}_2 + r\ \underline{W}_2$ →	↓ $\underline{V'}_2 = V'_1 + r\underline{V}_2$ $= V'_1 + rV_1$ ↓ $V'_2 = (1-r)\ (V'_1 + rV_1)$	↓ $\underline{W'}_2 = W'_1 + r\underline{W}_2$ $= W'_1 + r(W_1 + c\Delta t)$ ↓ $W'_2 = (1-r)(W'_1 + r(W_1 + c\Delta t))$	$a+c\Delta t$ a	$c\ \Delta t =$ $r\ \underline{V'}_2 + r\ \underline{W'}_2$ →
…									
$\boldsymbol{t =}$ $\mathbf{n}\ \boldsymbol{\Delta t}$		$\mathbf{V_n = (1-\boldsymbol{r})V_{n-1}}$ $\mathbf{= (1-\boldsymbol{r})^n \boldsymbol{a}}$				$\mathbf{V'_n = (1-\boldsymbol{r})(V'_{n-1} + \boldsymbol{r}V_{n-1})}$ $\mathbf{= n\ \boldsymbol{r}\ (1-\boldsymbol{r})^n \boldsymbol{a} = n\ \boldsymbol{r}\ V_n}$			

Initially, cask 1 only has wine, so $V_0 = a$, $W_0 = 0$, and cask 2 only has water, so $V'_0 = 0$, $W'_0 = a$. In the first interval the amount of water $c\Delta t$ flows into the first cask. So momentarily the interim wine $\underline{V}_0$ is still $V_0 = a$ and the interim water $\underline{W}_0$ is now $W_0+c\Delta t = c\Delta t$. The interim volume of cask 1 is now $a + c\Delta t$. But then an amount of liquid (wine and water) $c\Delta t$ flows out of cask 1 into cask 2, reducing the volume of cask 1 back to a.

As we did in the "Diluted Wine Puzzle", we shall consider fractions, that is,

$$c\Delta t = \frac{c\Delta t}{a+c\Delta t}(a+c\Delta t) = \frac{c\Delta t}{a+c\Delta t}(\underline{V}_1 + \underline{W}_1) = r(\underline{V}_1 + \underline{W}_1) = r\underline{V}_1 + r\underline{W}_1$$

where

$$r = \frac{c\Delta t}{a+c\Delta t}$$

This way we can treat the effects of the flow of the liquid separately for the wine and for the water, though we are only ultimately interested in the wine in cask 2. Now the remaining wine and water in cask 1 is $V_1 = (1-r)\underline{V}_1 = (1-r)V_0$ [$= (1-r)a$] and $W_1 = (1-r)\underline{W}_1 = (1-r)(W_0 + c\Delta t)$.

Still in the first time interval, we consider the effect of the flow of wine and water $c\Delta t$ from cask 1 into cask 2. The interim wine is now $\underline{V}'_1 = V'_0 + r\underline{V}_1 = V'_0 + rV_0$ and the interim water $\underline{W}'_1 = W'_0 + r\underline{W}_1 = W'_0 + r(W_0 + c\Delta t)$. Again the interim volume has increased to $a + c\Delta t$. But again an amount of liquid (wine and water) $c\Delta t = r\,(\underline{V}'_1+\underline{W}'_1)$ flows out of cask 2, reducing its volume back to a. The remaining wine and water in cask 2 is $V'_1 = (1-r)\underline{V}'_1 = (1-r)(V'_0 + rV_0)$ [$= (1-r)ra$] and $W'_1 = (1-r)\underline{W}'_1 = (1-r)(W'_0 + r(W_0 + c\Delta t))$.

This is the state of affairs at the end of the first interval. Proceeding sequentially and then unwinding the recursive relations, we have the amount of wine in the first cask at the end of n steps, or time $t = n\,\Delta t$, is

$$V_n = (1-r)V_{n-1} = (1-r)^n a$$

and the amount of wine in the second cask at the end of n steps is

$$V'_n = (1-r)(V'_{n-1} + rV_{n-1}) = n\,r\,(1-r)^n a = n\,r\,V_n$$

To obtain the continuous, instantaneous case at (fixed) time t, we need to take the limit as $\Delta t \to 0$ and $n = t/\Delta t \to \infty$. First, as $\Delta t \to 0$,

$$\boxed{nr = \frac{cn\Delta t}{a+c\Delta t} = \frac{ct}{a+c\Delta t} \to \frac{ct}{a}}$$

Now consider $(1-r)^n$.

$$(1-r)^n = \left(1-\frac{c\Delta t}{a+c\Delta t}\right)^{\frac{t}{\Delta t}} = \left(\frac{a}{a+c\Delta t}\right)^{\frac{t}{\Delta t}} = \left(\frac{1}{1+\frac{c}{a}\Delta t}\right)^{\frac{t}{\Delta t}} = \left(1+\frac{c}{a}\Delta t\right)^{-\frac{t}{\Delta t}} = \left((1+u)^{\frac{1}{u}}\right)^{-\frac{c}{a}t}$$

where $u = (c/a)\Delta t$. Therefore, as $\Delta t \to 0$, $u \to 0$ and $(1+u)^{1/u} \to e$. So as $\Delta t \to 0$ and $n = t/\Delta t \to \infty$, $(1-r)^n \to e^{-ct/a}$ and so

$$V_n = (1-r)^n a \;\to\; ae^{-\frac{ct}{a}}$$

and

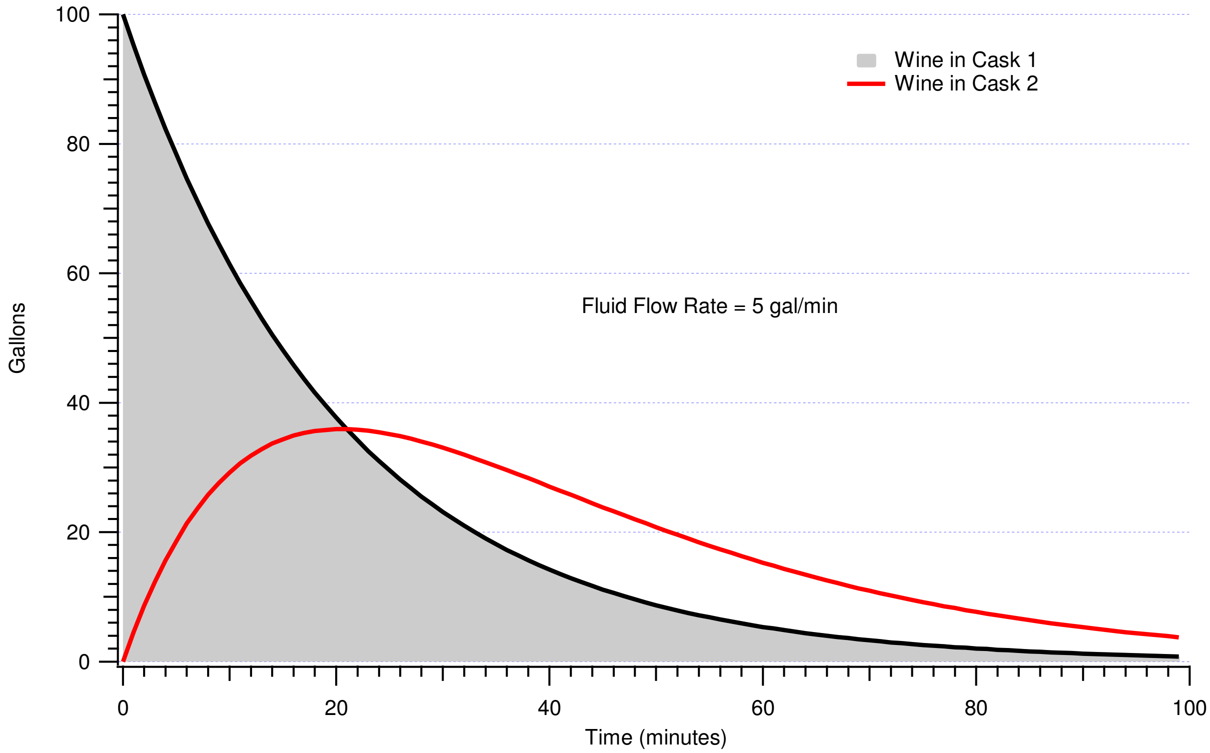


Figure 2 Time plots of wine in 100 gallon Casks 1 and 2 where the flow is 5 gallons/min

$$\boxed{V'_n = n\, r\, V_n \;\rightarrow\; cte^{-\frac{ct}{a}}}$$

This, then, is the expression for the wine in cask 2 at time t. I used Igor to calculate and plot the result (Figure 2). I took the case where the volume a of the casks was 100 gallons and the flow rate c was 5 gallons/min. For small t, the wine in cask 2 grows like $ct = 5t$, a straight line through the origin with slope 5. Then for large t, the exponential takes over and the curve looks like the exponential decay wine curve in cask 1. The wine in both casks eventually diminishes to nothing.

Bonus Problem

This is just a calculus problem in which we find the extremum of the differentiable function $V'(t) = cte^{-\frac{ct}{a}}$ by taking its derivative and setting it to zero. (Clearly from the characteristics of the function illustrated in Figure 2—or the behavior of the physical system—the extremum will be a maximum.)

$$\boxed{\frac{d}{dt}V' = \left(1 - \frac{c}{a}t\right)ce^{-\frac{ct}{a}} = 0 \;\Rightarrow\; t = a/c}$$

In the case of the Igor example above where a = 100 gallons and c = 5 gallons/min, we get the maximum occurs at $t = 100/5 =$ 20 minutes. And this maximum will be

$$5 \cdot 20 \cdot e^{-100/100} = 100/e = 36.79 \text{ gallons}$$

The Analyst Solution

I found a copy of the first year of *The Analyst* at JSTOR[2] from which I obtained their solution ([3]):

SOLUTION BY S. W. SALMON, MOUNT OLIVE, N. J.

Let Q be the quantity of wine in the lower cask at the end of t minutes; and let Q_1, Q_2, Q_3, … Q_n be the quantity at the end of the 1st, 2d, 3d, …, nth instants, q, the quantity of the mixture that escapes in an instant, and n, the number of instants in a minute, then is $n\,q = c$,[3]

$$Q_1 = q = q\left(1-\frac{q}{a}\right)^0,\ Q_2 = 2q\left(1-\frac{q}{a}\right),\ Q_3 = 3q\left(1-\frac{q}{a}\right)^2,\ \ldots,\ Q_n = nq\left(1-\frac{q}{a}\right)^{n-1}.$$

$$\therefore\ Q = tnq\left(1-\frac{q}{a}\right)^{tn-1}.$$ [4]

Taking the Napierian logarithm of both members of this equation,

$$\log Q = tn\log\left(1-\frac{q}{a}\right) + \log ct$$ [5]

$$= \log ct + tn\left(-\frac{q}{a} - \frac{q^2}{2a^2} - \frac{q^3}{3a^3} - \&c.\right).$$

Neglecting powers of infinitesimals,[6] we have

$$\log\left(\frac{ct}{Q}\right) = \frac{tnq}{a} = \frac{ct}{a}.$$

$$\therefore e^{\frac{ct}{a}} = \frac{ct}{Q}.\ \ \therefore Q = cte^{-\frac{ct}{a}};$$

where e denotes the base of the Napierian system of logarithms.

[This question was solved by Prof. Evans in an elegant manner by the method of finite differences, and in nearly the same manner as the above by Prof. Sensenig. All the other solutions were by application of the differential and integral calculus.]

It's difficult to describe "instantaneous" simultaneous actions without first considering discrete steps of "infinitesimal" duration Δt. This was not clearly explained, I felt, in *The Analyst* solution. I labored long and hard over my Table 1 to get the expressions just right (and it looks like S. W. Salmon made a mistake—the inclusion of –1 in the exponent—in jumping directly to a quasi-"instantaneous" formulation).

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2 https://www.jstor.org/stable/2636173?refreqid=excelsior%3A0cd8253f40f4a3ca5dbafc281992bbfd

3 JOS: I don't quite follow this; c is a rate of flow.

4 JOS: Q_n is the amount of wine after 1 minute. So to get the result after t minutes n is replaced by tn.

5 JOS: The "–1" in the previous exponent has disappeared in this step. This "error" increases the exponents in the expressions for Q_i which then correspond to my results. So the subsequent argument of essentially passing to the limit as $q \to 0$ ($\Delta t \to 0$) agrees with my answer. An alternative explanation for the missing exponent "–1" and consequent missing –log(1 – q/a) term is that in the limit, as $q \to 0$, –log(1 – q/a) $\to 0$ as well, and so doesn't contribute. Still, the derivation is a bit obscure to my mind.

6 JOS: Tantamount to saying $q \to 0$.

Bonus Problem

The Analyst provided two solutions to the Bonus Problem ([5]). One was the calculus solution I gave, but the other was different and involved interesting manipulations that probably could be justified rigorously.

SOLUTION BY PROF. D. M. SENSENIG, MILLERSVILLE, PA.

Let x represent the number of instants in a minute and u the quantity of wine in the lower cask at the end of t minutes, then will

$$u = tc\left(1-\frac{c}{ax}\right)^{tx-1},$$

which is maximum.

$$\frac{du}{dt} = \left[1 + tx\log'\left(1-\frac{c}{ax}\right)\right]\left(1-\frac{c}{ax}\right)^{tx-1} = 0$$ [7]

$$\therefore\ 1 + tx\log'\left(1-\frac{c}{ax}\right) = 0,$$

or $1 - tc/a = 0$, since $x = \infty$; whence $t = a/c$.

Note: $$0 = 1 + tx\log'\left(1-\frac{c}{ax}\right) = 1 + t\ln\left(1-\frac{c}{ax}\right)^{x} = 1 + t\ln\left[\left(1+\left(-\frac{c}{ax}\right)\right)^{-\frac{ax}{c}}\right]^{-\frac{c}{a}} \to 1 - \frac{tc}{a}\ln(e) = 1 - \frac{tc}{a},$$

as $x \to \infty$

This approach, essentially involving the limits on two variables t and x (since the derivative with respect to t involves a limit on the variable t—*while x is assumed constant*), asserts that the limit of the maximum value and maximum point of approximating functions equals the maximum value of the limit of those functions at the limit of the maxima points. Double limits can be tricky, especially involving sequences of functions, but probably justified in this case.

In solving the above problem, I observed the following beautiful law: Suppose the number of casks containing water to be indefinitely increased, and the other conditions remain the same. Develop into a series the expression

$$\left[\left(1-\frac{c}{ax}\right)+\frac{c}{ax}\right]^{tx},$$

in which x represent the number of instants in a minute, thus:

$$\left[\left(1-\frac{c}{ax}\right)+\frac{c}{ax}\right]^{tx} = \left(1-\frac{c}{ax}\right)^{tx} + tx\left(1-\frac{c}{ax}\right)^{tx-1}\frac{c}{ax} + \frac{tx(tx-1)}{2}\left(1-\frac{c}{ax}\right)^{tx-2}\frac{c^2}{a^2x^2} + etc.$$

If we now represent the parts of wine in each cask at the end of t minutes, beginning with the first, respectively by u, u', u'', u''', etc., then will

[7] JOS: Apparently "log′ " is the same as the natural or Napierian log "ln". Sensenig has also dropped the constant c from the derivative equation.

$$u=\left(1-\frac{c}{ax}\right)^{tx}\text{, a minimum when } t=\infty$$

$$u'=tx\left(1-\frac{c}{ax}\right)^{tx-1}\frac{c}{ax}\text{, a maximum when } t=a/c$$

$$u''=\frac{tx(tx-1)}{2}\left(1-\frac{c}{ax}\right)^{tx-2}\frac{c^2}{a^2x^2}\text{, a maximum when } t=2a/c$$

$$u'''=\frac{tx(tx-1)(tx-2)}{3\cdot 2}\left(1-\frac{c}{ax}\right)^{tx-3}\frac{c^3}{a^3x^3}\text{, a maximum when } t=3a/c\text{, etc., etc., etc.}$$

The values of u, u', u'', u''', etc., may readily be found by developing the above expressions by the binomial theorem and substituting $x = \infty$ or by applying the principles of logarithms as the student may prefer.

I leave it as an exercise to the reader to justify these last statements.

Comment

By the way, since I have been rooting around an American publication of math problems in the 19th century, I found the Americans were just as wed to brutal computations as the British in their 19th century journals of math problems. For example, on the same page as the above solution to the Bonus Problem was given, some new problems were posed. Here are the first two:

25. By R. L. SELDON, Troy, N. Y.—Required the sides of an obtuse angled triangle the area of which is 14.048 acres, the obtuse angle 111° 15′, and one of the acute angles 11° 44′ 10″.

26. By WM. HOOVER, South Bend, Ind.—Find θ from the equation

$$15 \sin \theta + 12 \cos \theta = 17.97240 \text{ .... (1).}$$

This reminds me of the agonizing problems I had in my high school trigonometry class, which required the use of log and trig tables to compute (using the Babylonian sexagesimal number system no less!). As far as I am concerned, the greatest by-product of the 20th century space program was the hand-held calculator. I remember in the 1970s, when they first came out, being in a used-book store and seeing a book of log and trig tables. After almost 400 years I realized it had been rendered *instantly* obsolete. Wow! And now I have the power of Igor at my fingertips and it is fantastic. For all the agonies of the current times, there is still joy to be found.

References

[1] Wardhaugh, Benjamin, ed., *A Wealth of Numbers: An Anthology of 500 Years of Popular Mathematics Writing*, Princeton University Press, Princeton, New Jersey, 392pp., 2012. pp.71-72

[2] *The Analyst: A monthly journal of pure and applied mathematics*, Vol. 1, No. 1 (Des Moines) (Jan., 1874), p. 15.

[3] ———, Vol. 1, No. 3 (Mar., 1874), pp. 50-51

[4] ———, Vol. 1, No. 4 (Apr., 1874), p. 71

[5] ———, Vol. 1, No. 6 (Jun., 1874), pp. 111-112