Three Equal Circles

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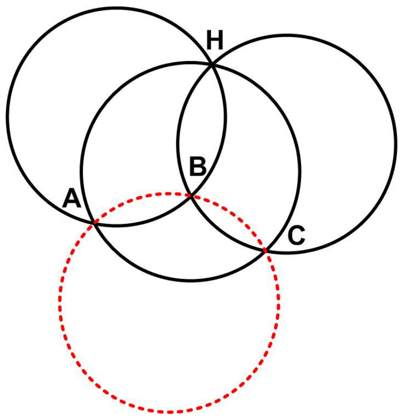

Here is a problem from the *Quantum* magazine ([1]), only this time from the "Challenges" section (these are expected to be a bit more difficult than the Brainteasers).

Three circles with the same radius r all pass through a point H. Prove that the circle passing through the points where the pairs of circles intersect (that is, points A, B, and C) also has the same radius r.

Indeed, I found this quite challenging. It took me several weeks to work out my approach and details.

My Solution

Without loss of generality we can assume the outer two circles (green in Figure 1) are horizontally spaced an arbitrary distance apart so that they still intersect. We then consider the third (blue) circle intersecting the top intersection of the two green circles (H) and pivoting an arbitrary amount about this point. This covers all cases in the problem.

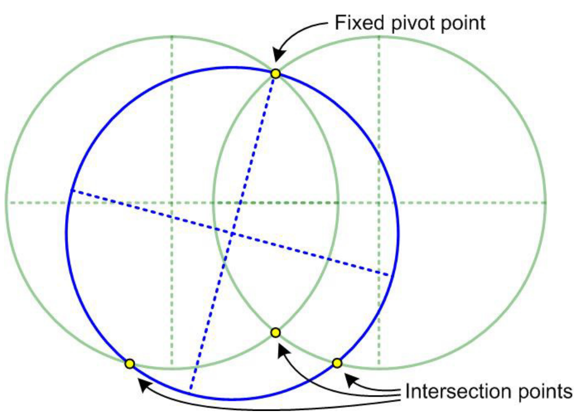

Figure 1 Problem Statement

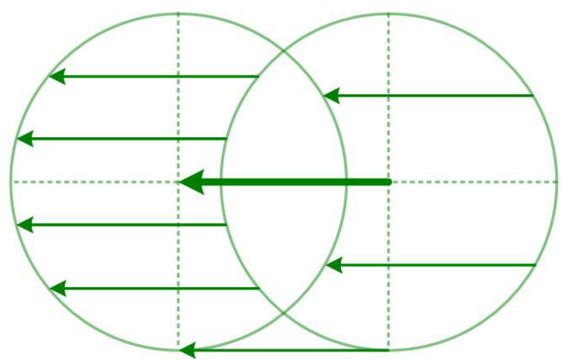

Figure 2 Translating Two Circles

Now note concerning the pair of green circles that the distance separating their centers is also the distance each point on the right-hand circle is translated horizontally to its corresponding point on the left-hand circle.

We come now to the crucial facts of the problem. Position the third blue circle symmetrically with respect to the two green circles as shown in Figure 3. Draw two straight lines from the top intersection/pivot point to the centers of the two green circles. These lines are equal to the equal radius r of the two green circles, and so form an isosceles (green) triangle with a base the separation of the two centers. Now translate an image of this triangle (blue) vertically down a distance r so that its two base vertices are the bottom points of the two green circles and its top vertex lies on the center of the blue circle. It is essential to understand that because the top vertex of the blue triangle lies at the center of the blue circle and the sides of the blue triangle are also radii of the blue circle, these

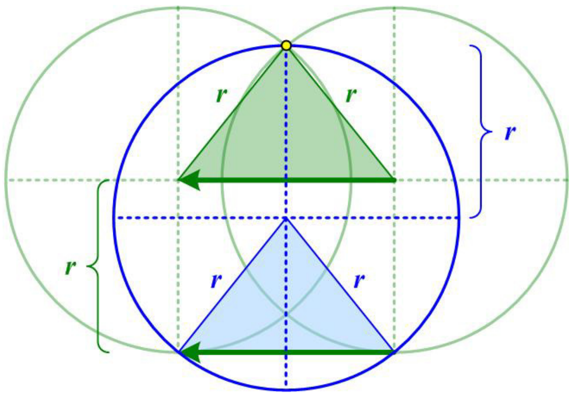

Figure 3 Third Circle Initial Position

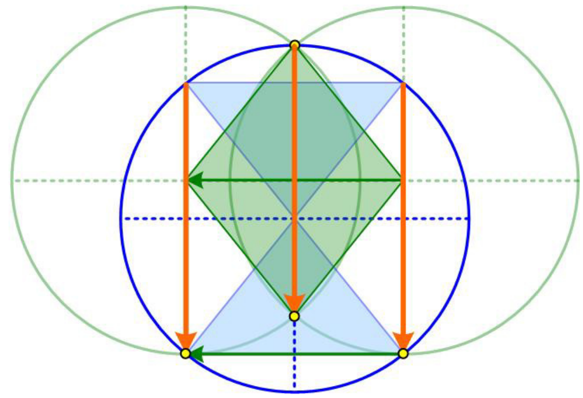

Figure 4 Third Circle Initial Position

base vertices are also intersection points of the blue circle with the two green circles (and separated horizontally by the distance between the green circles).

The second important fact is shown in Figure 4. By flipping the green and blue triangles as shown, we see the (orange) vertical distance between the points of the blue circle lying directly over the intersection points are equal to twice the altitudes of the triangles, and both these are equal to the (orange) distance between the top and the bottom intersection points of the two green circles.

Now Figure 5 shows what happens when the blue circle is rotated arbitrarily around the top pivot point: its intersections with the two green circles remain horizontally separated by the constant green circle separation distance and the three (orange) vertical distances remain constant.

All this means we can translate down vertically an (orange) image of the blue circle a distance equal to the (orange) separation between the intersection points of the green circles. The resulting orange circle will pass through the three intersection points of the blue and green circles and, being an image of the blue circle, has the same radius as the other three circles. Since three points determine a circle, there can be no other sized circle through these three points, and this proves the desired statement.

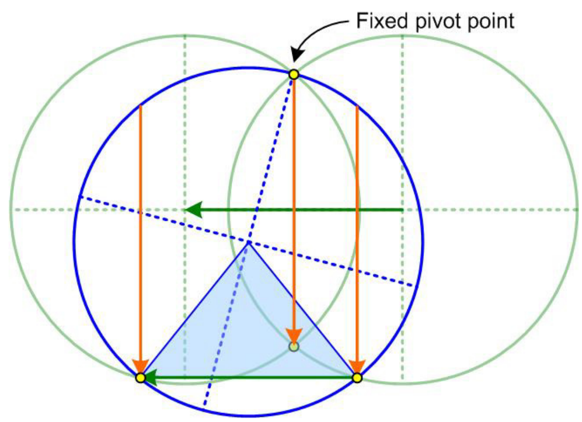

Figure 5 Third Circle Pivot

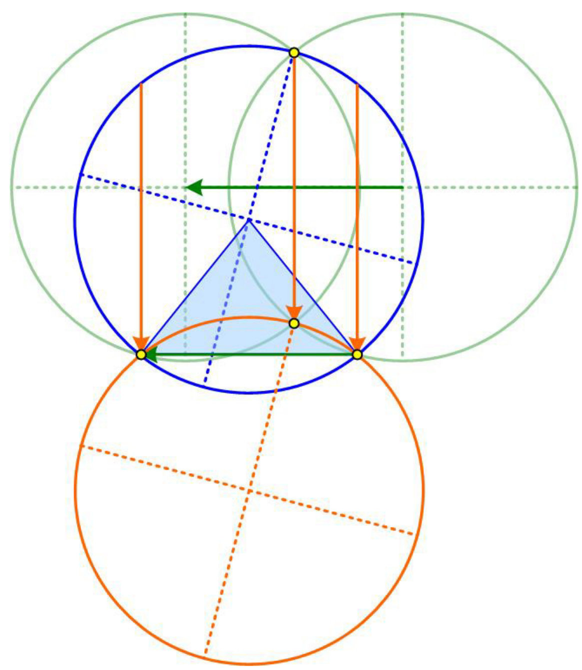

Figure 6 Solution

Quantum Solution

As usual, the *Quantum* solution is slicker and briefer, though I have to admit, I don't see how they verified some of their steps.

For each of the three given circles, draw radii from its center to the points where it intersects the other two (the thin lines to points A, B, C, H in figure 1). Three rhombi appear (with common vertex H) whose sides are all equal to r. We can imagine them as representing three faces of a parallelepiped with common vertex H. Draw the three other faces—that is, three new rhombi with common vertex O. Their new sides (the thick lines OA, OB, OC in figure 1) are also equal to r.[1] And this is precisely what has to be proved. (Notice that our reasoning also holds when B lies outside triangle AHC.) Figure 2 creates a more convenient view of the situation by breaking apart the parallelepiped and eliminating the distracting portions of figure l.

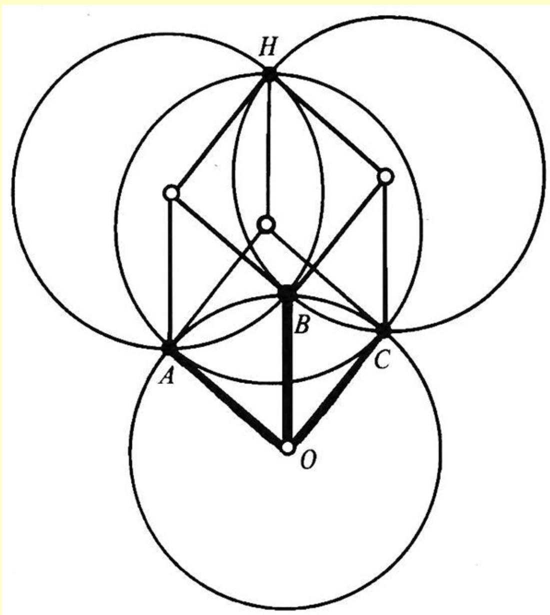

Figure 7

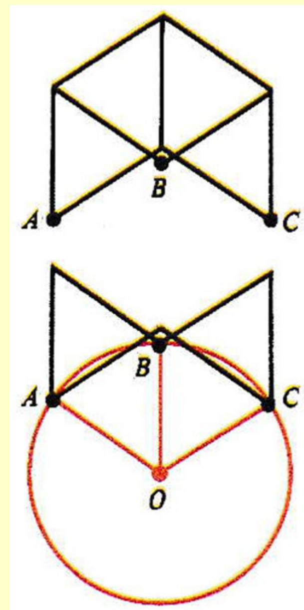

Figure 8

We'll leave it to you to find other solutions to this problem and lots of interesting facts of triangle geometry related to it. You may have noticed, for example, that triangle ABC is congruent to the triangle with vertices at the centers of the three given circles; that B is the orthocenter of triangle AHC; that the points symmetrical to B relative to the sides of triangle AHC lie on its circumcircle; and that all four circumferences play equivalent roles—that is, every group of three circles has a common point.

You can find a detailed discussion of this problem in the remarkable book *Mathematical Discovery* by George Polya.

References

[1] "Three Equal Circles" M6 "Challenges" *Quantum Magazine*, P2, National Science Teachers Assoc., Springer-Verlag, May 1990, p.22

[1] JOS: Why? Is OB parallel to the vertical radii r? Why? I agree this is the crux of the problem, but I don't see how they proved it? Just *calling* them faces of a parallelepiped doesn't make them faces. And what does a 3-dimensional object have to do with a 2-dimensional problem? But I guess I am just being dense.