# Pole Leveling Puzzle

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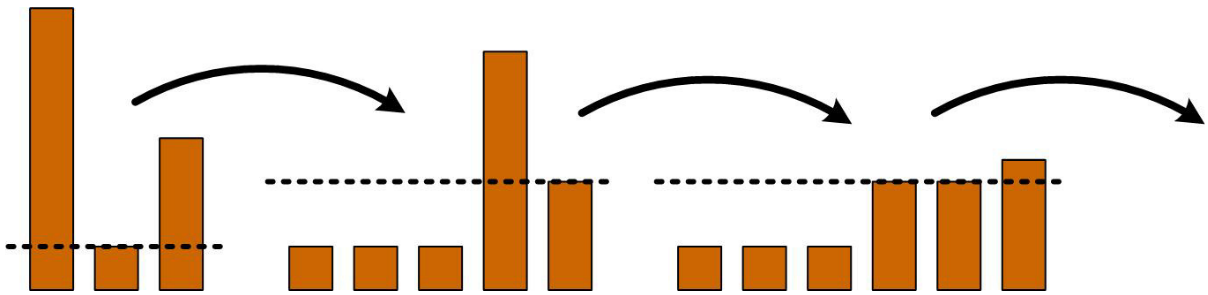

This is another thoughtful puzzle from the imaginative mind of James Tanton (with slight edits).[1]

Three poles of height 1183 feet, 182 feet, 637 feet stand in the ground. Pick a pole and saw off all the taller poles at that height. Plant those tops in the ground too. Repeat until no more such saw cuts can be made. Despite choices made along the way, what final result is sure to occur? [Four poles, heights a, b, c, d ft?]

## Solution

The process stops when all the poles are the same height. Since we haven't thrown away any of the pole material (assume clean saw cuts!), when we add up all the pieces we should get the sum of the original pole lengths. That is, each of the poles will be represented by some integer multiple of the final pole height. Thus this final pole height is the greatest common divisor of the heights of the three original poles. (So the answer to the general four pole problem is GCD(a, b, c, d).)

In our problem, we have poles of length

$$1183 = 7{\bullet}13^2 = 13{\bullet}91$$

$$182 = 2{\bullet}7{\bullet}13 = 2{\bullet}91$$

$$637 = 7^2{\bullet}13 = 7{\bullet}91$$

Therefore, we will finally end up with 2 + 7 + 13 = 22 poles of height 91 feet.

Notice that every subtraction of a pole height involves subtracting a multiple of 91 from other multiples of 91, so we are always left with multiples of 91 until the multiple is 1. In the picture we have

**Step 1: Subtract 182 pole**

$1183 - 182 = 13{\bullet}91 - 2{\bullet}91 = 11{\bullet}91 = 1001$ (1 time)

$637 - 182 = 7{\bullet}91 - 2{\bullet}91 = 5{\bullet}91 = 455$ (1 time)

$182 = 2{\bullet}91$ (1 + 2 = 3 times)

**Step 2: Subtract 455 pole**

$1001 - 455 = 11{\bullet}91 - 5{\bullet}91 = 6{\bullet}91 = 546$ (1 time)

$455 = 5{\bullet}91$ (1 + 1 = 2 times)

[1] https://twitter.com/jamestanton/status/1342830242579140608

| | |
|---|---|
| $182 = 2{\cdot}91$ | (3 times) |

**Step 3: Subtract 455 pole again**

| | |
|---|---|
| $546 - 455 = 6{\cdot}91 - 5{\cdot}91 = 1{\cdot}91 = 91$ | (1 time) |
| $455 = 5{\cdot}91$ | (2 + 1 = 3 times) |
| $182 = 2{\cdot}91$ | (3 times) |

**Step 4: Subtract 182 pole again**

| | |
|---|---|
| $455 - 182 = 5{\cdot}91 - 2{\cdot}91 = 3{\cdot}91 = 273$ | (3 times) |
| $182 = 2{\cdot}91$ | (3 + 3 = 6 times) |
| $91 = 1{\cdot}91$ | (1 time) |

**Step 5: Subtract 182 pole again**

| | |
|---|---|
| $273 - 182 = 3{\cdot}91 - 2{\cdot}91 = 91$ | (3 times) |
| $182 = 2{\cdot}91$ | (6 + 3 = 9 times) |
| $91 = 1{\cdot}91$ | (1 time) |

**Step 6: Subtract 91 pole**

| | |
|---|---|
| $182 - 91 = 2{\cdot}91 - 1{\cdot}91 = 91$ | (9 times) |
| 91 | (4 + 9 = 13 times) |
| | Total = 9 + 13 = 22 times |

# Comment

It has been so long that I had forgotten the methods for computing the GCD of numbers. A visit to *Wikipedia*[2] showed that the pole leveling method was really an example of the **Euclidean Algorithm** for computing the GCD:

> The Euclidean algorithm is based on the principle that the greatest common divisor of two numbers does not change if the larger number is replaced by its difference with the smaller number. For example, 21 is the GCD of 252 and 105 (as $252 = 21 \times 12$ and $105 = 21 \times 5$), and the same number 21 is also the GCD of 105 and $252 - 105 = 147$ [$= 21 \times 7$]. Since this replacement reduces the larger of the two numbers, repeating this process gives successively smaller pairs of numbers until the two numbers become equal. When that occurs, they are the GCD of the original two numbers.

So James Tanton has posed a clever concrete problem to illustrate an abstract mathematical algorithm. This is the sort of trick Feynman would pull—posing a concrete problem whose solution comes from the application of an abstract idea. Sometimes physicists and mathematicians spend so much time in the abstract realm that they forget how to recognize what is happening in concrete situations.



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[2] See https://en.wikipedia.org/wiki/Euclidean_algorithm and see also https://en.wikipedia.org/wiki/Greatest_common_divisor