Playing with Polys

7 November 2020

Jim Stevenson

thumbs.dreamstime.com

Here is a fairly straight-forward problem from *500 Mathematical Challenges* ([1]).

Problem 256. Let n be a positive integer. Show that $(x - 1)^2$ is a factor of $x^n - n(x - 1) - 1$.

Solution

First,

$$\begin{aligned} x^n - n(x-1) - 1 &= (x^n - 1) - n(x-1) \\ &= (x-1)(x^{n-1} + x^{n-2} + x^{n-3} + \ldots + x + 1) - n(x-1) \\ &= (x-1)(x^{n-1} + x^{n-2} + x^{n-3} + \ldots + x + 1 - n) \qquad (1) \\ &= (x-1)\,p(x) \end{aligned}$$

Next, recall the result from abstract algebra that if a is a root of the polynomial equation $p(x) = 0$, then $(x - a)$ is a factor of the polynomial $p(x)$.[1] Now from equation (1) $p(1) = 0$. Therefore there is some polynomial $m(x)$ such that

$$p(x) = (x-1)\,m(x) \;\Rightarrow\; x^n - n(x-1) - 1 = (x-1)^2\,m(x)$$

which shows that $(x - 1)^2$ is a factor of $x^n - n(x - 1) - 1$.

References

[1] Barbeau, Edward J., Murray S. Klamkin, William O. J. Moser, *Five Hundred Mathematical Challenges*, Spectrum Series, Mathematical Association of America, Washington D.C, 1995

[1] The division algorithm says in general if $q(x)$ is a polynomial of degree less than or equal to the degree of $p(x)$, then we can divide $p(x)$ by $q(x)$ to get polynomials $m(x)$ and $r(x)$ such that $p(x) = m(x)\,q(x) + r(x)$ where $\deg r(x) < \deg q(x)$. Therefore setting $q(x) = (x - a)$ we can write $p(x) = m(x)\,(x - a) + r(x)$ where $0 = \deg r(x) < \deg (x - 1) = 1$, so $r(x)$ is a constant. But if $x = a$ is a root, then $0 = p(a) = m(x)\cdot 0 + r$. So $r = 0$ and $p(x) = m(x)\,(x - a)$, that is, $(x - a)$ is a factor of $p(x)$.