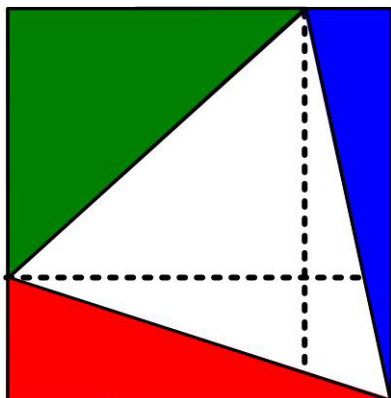


Diabolical Triangle Puzzle

28 January 2021

Jim Stevenson



This simple-appearing problem is from the 17 August 2020 MathsMonday offering¹ by MEI, an independent curriculum development body for mathematics education in the UK.

The diagram shows an equilateral triangle in a rectangle. The two shapes share a corner and the other corners of the triangle lie on the edges of the rectangle. Prove that the area of the green triangle is equal to the sum of the areas of the blue and red triangles. What is the most elegant proof of this fact?

Since the MEI twitter page seemed to be aimed at the high school level and the parting challenge seemed to indicate that there was one of those simple, revealing solutions to the problem, I spent several *days* trying to find one. I went down a number of rabbit holes and kept arriving at circular reasoning results that assumed what I wanted to prove. Visio revealed a number of fascinating relationships, but they all assumed the result and did not provide a proof. I finally found an approach that I thought was at least semi-elegant.

My Solution

I tried a number of computational approaches, but they got too complicated and fraught with mistakes. Moreover, in the formulations I was using I was having a difficult time relating the blue and red triangles to the green triangle. Finally I resorted to vectors, which captured a way to relate the edges of the equilateral triangle with the edges of the rectangle via their horizontal and vertical components.

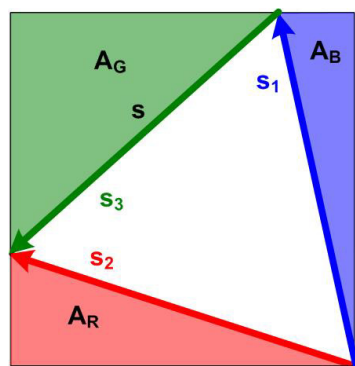


Figure 1

Let the areas of the three colored triangles be given by A_G , A_B , A_R as in Figure 1. Let s_k for $k = 1, 2, 3$, be the vectors of length s lying along each of the sides of the equilateral triangle. Figure 2 shows the key insight, that the areas of interest are between the side vectors and their horizontal components.

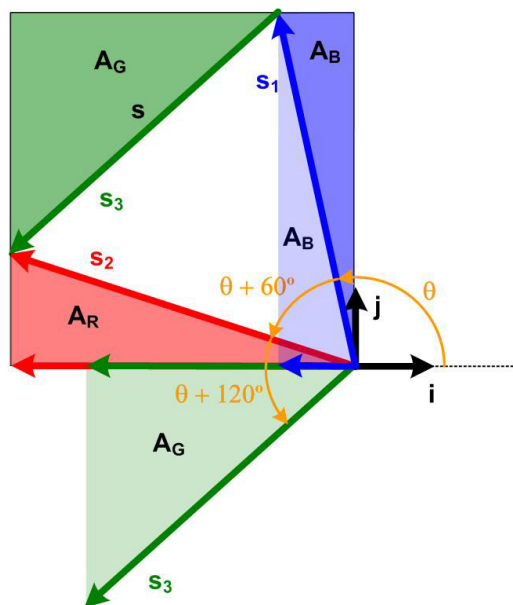


Figure 2

¹ <https://twitter.com/MEIMaths/status/1295284235599859713>

So for $k = 1, 2, 3$, the side vectors are given by

$$\mathbf{s}_k = s \cos(\theta + (k-1)60^\circ) \mathbf{i} + s \sin(\theta + (k-1)60^\circ) \mathbf{j}$$

Then horizontal components are the projections onto the horizontal axis, namely, for $k = 1, 2, 3$,

$$\mathbf{s}_k \cdot \mathbf{i} = s \cos(\theta + (k-1)60^\circ)$$

Using the vector cross product, the areas are

$$A_B = \frac{1}{2} \mathbf{s}_1 \times (\mathbf{s}_1 \cdot \mathbf{i}) \mathbf{i}, \quad A_R = \frac{1}{2} \mathbf{s}_2 \times (\mathbf{s}_2 \cdot \mathbf{i}) \mathbf{i}, \quad A_G = \frac{1}{2} (\mathbf{s}_3 \cdot \mathbf{i}) \mathbf{i} \times \mathbf{s}_3 = -\frac{1}{2} \mathbf{s}_3 \times (\mathbf{s}_3 \cdot \mathbf{i}) \mathbf{i}$$

where we have reversed the order of the last product to maintain the same orientation of the angles between the vectors. Now for $k = 1, 2, 3$, the cross products are given by

$$\mathbf{s}_k \times \mathbf{i} = s \sin(\theta + (k-1)60^\circ) \mathbf{j} \times \mathbf{i} = -s \sin(\theta + (k-1)60^\circ) \mathbf{k}$$

Taking just the scalar component of the vector products and doubling the results, we have

$$2A_B = \mathbf{s}_1 \times (\mathbf{s}_1 \cdot \mathbf{i}) \mathbf{i} = (\mathbf{s}_1 \cdot \mathbf{i}) (\mathbf{s}_1 \times \mathbf{i}) = -s^2 \cos \theta \sin \theta = -\frac{1}{2} s^2 \sin 2\theta$$

$$2A_R = (\mathbf{s}_2 \cdot \mathbf{i}) (\mathbf{s}_2 \times \mathbf{i}) = -s^2 \cos(\theta + 60^\circ) \sin(\theta + 60^\circ) = -\frac{1}{2} s^2 \sin 2(\theta + 60^\circ)$$

$$2A_G = -(\mathbf{s}_3 \cdot \mathbf{i}) (\mathbf{s}_3 \times \mathbf{i}) = s^2 \cos(\theta + 120^\circ) \sin(\theta + 120^\circ) = \frac{1}{2} s^2 \sin 2(\theta + 120^\circ)$$

Therefore, using Figure 3

$$4A_B = -s^2 \sin 2\theta$$

$$4A_R = -s^2 \sin(2\theta + 120^\circ) \\ = -s^2 [\sin 2\theta (-1/2) + \cos 2\theta (\sqrt{3}/2)]$$

$$4A_G = s^2 \sin(2\theta + 240^\circ) \\ = s^2 [\sin 2\theta (-1/2) + \cos 2\theta (-\sqrt{3}/2)]$$

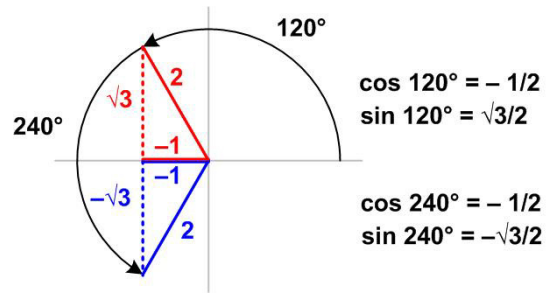


Figure 3

And from this we get our proof:

$$A_G = A_B + A_R$$

This approach does involve some trigonometry, but it is relatively straight-forward, though the proliferation of negative signs requires careful attention to get the answer to come out right. From a geometrical point of view the vector approach is clarifying, since it shows that as θ varies, moving vector \mathbf{s}_1 , the other two vectors move in lock-step at 60° intervals, thus tying all the colored triangles together.

Rabbit Hole. Since I spent so much time on other approaches, I thought I would show the error of my ways in one of them.

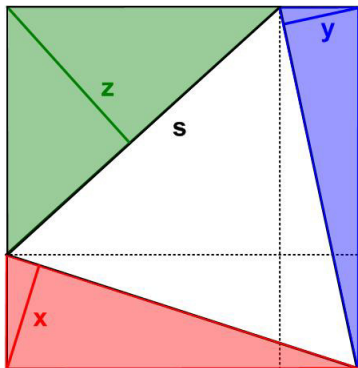


Figure 4

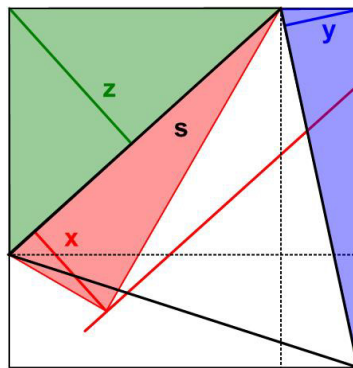


Figure 5

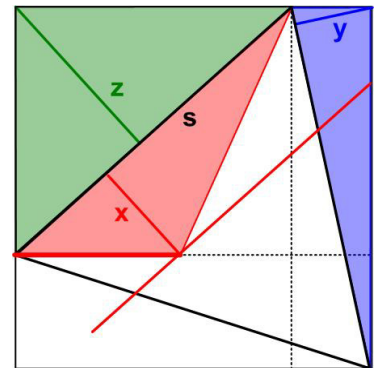


Figure 6

The idea was to use geometric transformations to fill the duplicate image of the green triangle inside the equilateral triangle with the red and blue triangles. I indicated the altitudes x , y , z of the three triangles and the length s of the equilateral triangle (Figure 4). Then I rotated the red triangle 60° to lie against the green triangle along the side s (Figure 5). Using the area-preserving shear method, I moved the vertex of the red triangle along the parallel red line until it intersected the horizontal dashed line (Figure 6).

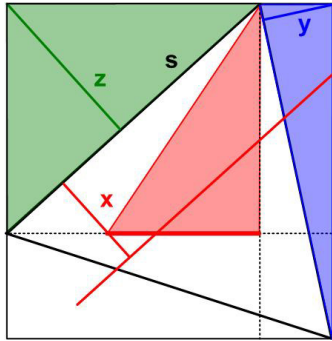


Figure 7

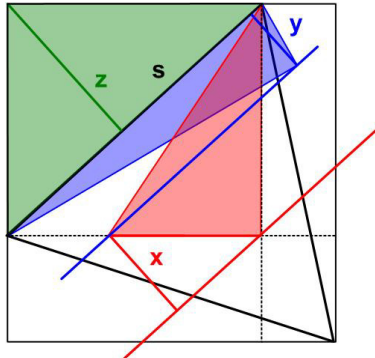


Figure 8

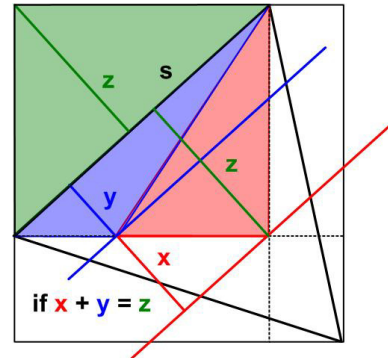


Figure 9

I then slid the base of the red triangle horizontally until it touched the vertical dashed line (Figure 7). I moved the red parallel line down until it intersected the vertex of the red triangle and rotated the blue triangle 60° clockwise to lie against the green triangle along the side s (Figure 8). Amazingly, the blue parallel line just intersected the end of the x altitude. I moved the vertex of the blue triangle along the blue line until it intersected the dashed horizontal line (Figure 9), and voila, the red and blue triangles filled the area of the green triangle.

However, as Figure 9 shows, the last two operations were tantamount to assuming $x + y = z$. Just because Visio showed it was so, that was not a proof. In fact, since $A_G = sz$, $A_B = sy$, and $A_R = sx$, the equation $x + y = z$ is equivalent to $A_G = A_B + A_R$, which was what I wanted to prove. To make the diagram work, I was tacitly assuming the answer I was trying to prove!

MEI Solutions

I did not find particularly elegant solutions to the problem at the MEI twitter page. I had tried some of these direct approaches as well, but my arithmetic failed me.

Nick Kalapodis² Solution

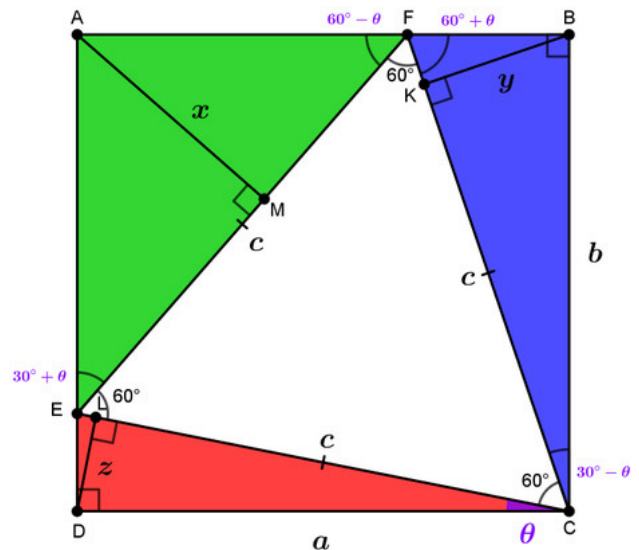
Let a , b be the dimensions of the rectangle and c be the side length of the equilateral triangle. We set $\theta = \angle ECD$ and let x , y , z be the heights of the triangles. Then

$$z = \sin \theta \cdot a = \sin \theta \cos \theta \cdot c$$

$$y = \sin (30^\circ - \theta) \cdot b = \sin (30^\circ - \theta) \sin (30^\circ + \theta) \cdot c$$

and

$$c = EM + MF = x \cdot \cot (30^\circ + \theta) + x \cdot \cot (60^\circ - \theta)$$



² 7:35 PM · Aug 19, 2020 <https://twitter.com/NickKalapodis/status/1296229357682397189>

$$\Rightarrow x = c / (\cot(30^\circ + \theta) + \cot(60^\circ - \theta))$$

So it suffices to prove that

$$x = y + z$$

$$\frac{1}{\cot(30^\circ + \theta) + \cot(60^\circ - \theta)} = \frac{\sin(30^\circ - \theta) \sin(30^\circ + \theta) + \sin \theta \cos \theta}{\sin(30^\circ - \theta) \sin(30^\circ + \theta) + \sin \theta \cos \theta}$$

$$\frac{2}{\cot(30^\circ + \theta) + \tan(30^\circ + \theta)} = \frac{2 \sin(30^\circ - \theta) \sin(30^\circ + \theta) + 2 \sin \theta \cos \theta}{\sin(30^\circ - \theta) \sin(30^\circ + \theta) + \sin \theta \cos \theta}$$

$$\frac{2 \sin(30^\circ + \theta) \cos(30^\circ + \theta)}{\cos^2(30^\circ + \theta) + \sin^2(30^\circ + \theta)} = \frac{\sin(60^\circ - 2\theta) + \sin 2\theta}{\sin(60^\circ - 2\theta) + \sin 2\theta}$$

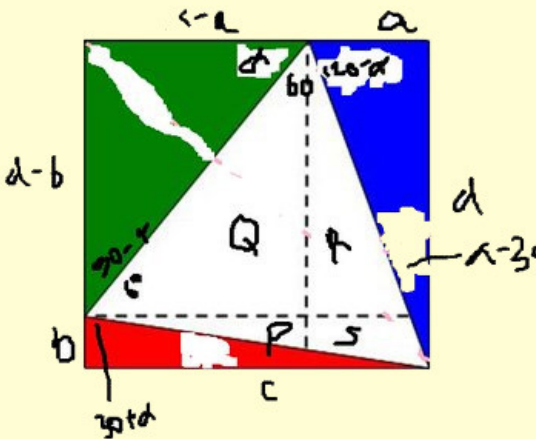
$$\sin(60^\circ + 2\theta) = \sin(60^\circ - 2\theta) + \sin 2\theta$$

$$\sin(60^\circ + 2\theta) - \sin(60^\circ - 2\theta) = \sin 2\theta$$

$$2 \cos 60^\circ \sin 2\theta = \sin 2\theta$$

which is true.

Todi Nagawe³ Solution



$$\begin{aligned} \sin[(a+30)+(a-30)] &= \sin(a+30)\cos(a-30) + \cos(a+30)\sin(a-30) \\ \sin(2a) &= c/s \cdot d/s + b/s \cdot a/s \\ 2\sin(a)\cos(a) &= (cd+ab)/s^2 \\ 2(d-b)/s \cdot (c-a)/s &= (cd+ab)/s^2 \\ 2(cd-ad-bc+ab) &= cd+ab \\ 2cd-2ad-2bc+2ab-cd-ab &= 0 \\ cd-2ad-2bc+ab &= 0 \\ cd+ab &= 2ad+2bc \\ P+Q+R+S+S &= 2R+2S+2P+2S \\ Q &= R+2S+P \\ Q &= (R+S)+(S+P) \\ 2gr &= 2blu+2red \\ gr &= blu+red \end{aligned}$$

Robert Forrester⁴ Solution

Adjust the diagram, so the red triangle disappears, and we have a symmetrical arrangement, where green = blue. The triangle and square then share the same lower side. Do the same and make the blue triangle disappear. Then green = red. :)

Clearly this last solution is only a partial one, since just two extreme cases were considered.

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³ 10:34 AM · Aug 22, 2020 <https://twitter.com/TodiNagawe/status/1297180223616630785>

⁴ 6:48 AM · Oct 9, 2020 <https://twitter.com/RAnForr784/status/1314518110599249922>