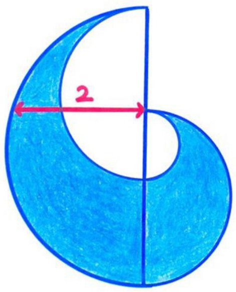
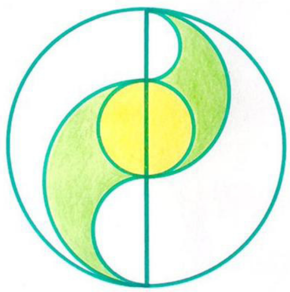
Geometric Puzzle Marvels

21 November 2020

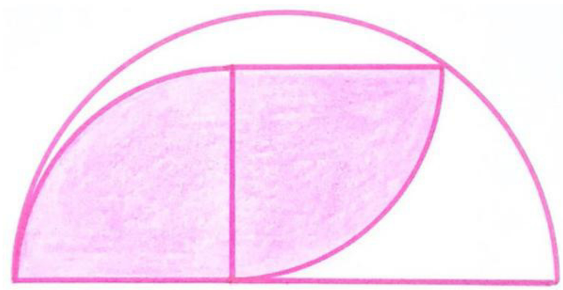
Jim Stevenson



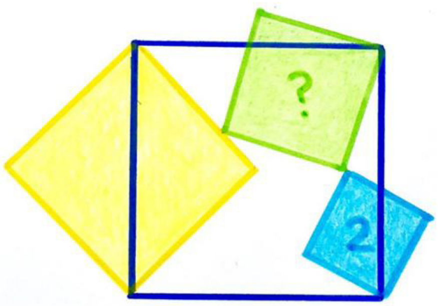
#1 Largest and smallest semicircles are concentric. What's the shaded area?



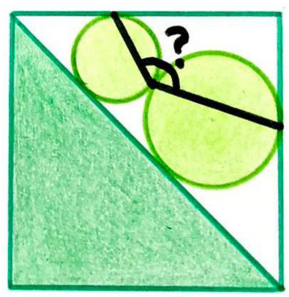
#2 The green area is double the yellow area. What fraction of the total is shaded?



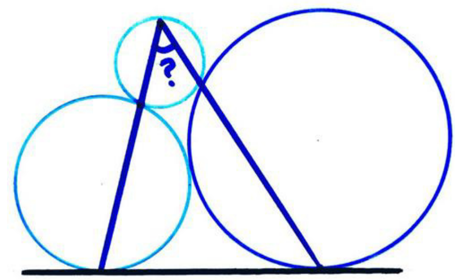
#3 What fraction of the semicircle do these quarter circles cover?



#4 Four squares. The blue area is 2. What's the green area?



#5 Two circles in a square. What's the angle?



#6 The circles have diameters 1, 2 and 3. What's the angle?

Here is another collection of beautiful geometric problems from Catriona Agg (née Shearer). They never fail to brighten the day with their loveliness.

Solution to Problem #1[1]

The first thing we do is rotate the lower right semicircle 180° as shown in Figure 1 and Figure 2. Thus the blue area is $\pi(R^2 - r^2)/2$, where R is the radius of the large semicircle and r is the radius of the small semicircle. As shown in Figure 2, we can inscribe a right triangle in the large semicircle and obtain the geometric mean relation

$$\frac{R+r}{2} = \frac{2}{R-r}$$

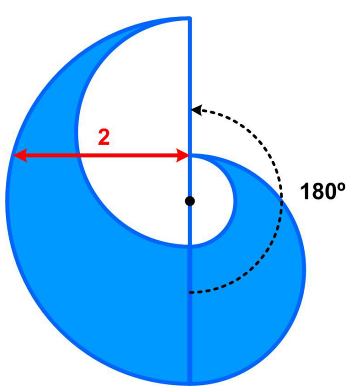


Figure 1

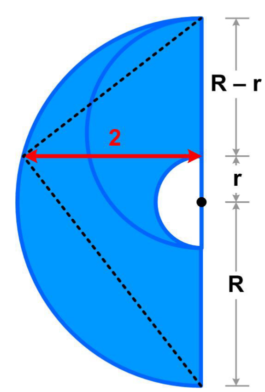


Figure 2

[1] 5:51 AM • Jun 14, 2020 https://twitter.com/Cshearer41/status/1272104391567388672

Therefore $R^2 - r^2 = 4$ and the area of the blue region is then 2π. It is interesting that the actual values for the radii are not needed.

In general, it is possible to go to the original Twitter site for the problem to see other solutions. For example, the geometric mean can be avoided and the Pythagorean Theorem used instead, since $R^2 = 2^2 + r^2$ yields $R^2 - r^2 = 4$ immediately. (I didn't notice this at first.)

Solution to Problem #2[2]

Using Figure 3, and letting G be the green area and Y the yellow area, we have G = 2Y where

$$G = [\tfrac{1}{2}\pi(r_2 + r_3)^2 - \tfrac{1}{2}\pi r_2^2 - \tfrac{1}{2}\pi r_3^2] + [\tfrac{1}{2}\pi(r_1 + r_2)^2 - \tfrac{1}{2}\pi r_1^2 - \tfrac{1}{2}\pi r_2^2]$$

$$= \pi r_2 (r_1 + r_3)$$

and

$$Y = \pi r_2^2$$

So G = 2Y implies $2r_2 = r_1 + r_3$. Therefore $R = 3r_2$ and the ratio of the shaded area to the large circle is

$$\frac{G+Y}{\pi R^2} = \frac{3Y}{\pi R^2} = \frac{3\pi r_2^2}{9\pi r_2^2} = \frac{1}{3}$$

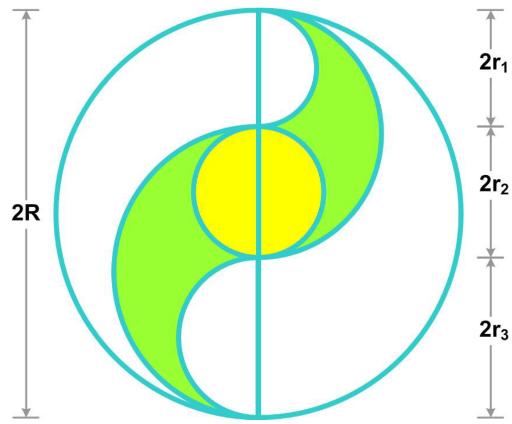


Figure 3

Solution to Problem #3[3]

Using the geometric mean as shown in Figure 4, we have

$$\frac{2r}{r} = \frac{r}{2R - 2r}$$

Therefore

$$\frac{r}{R} = \frac{4}{5}$$

which implies the ratio of the area of the pink region to the entire semicircle is

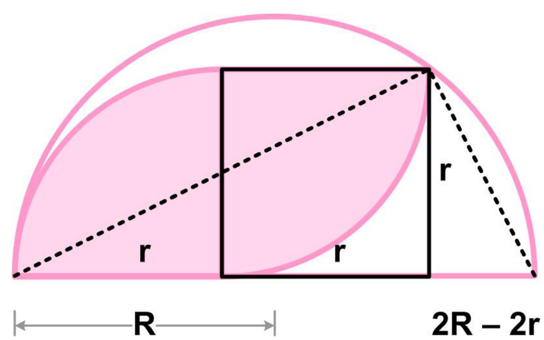


Figure 4

$$\frac{\frac{1}{2}\pi r^2}{\frac{1}{2}\pi R^2} = \left(\frac{r}{R}\right)^2 = \left(\frac{4}{5}\right)^2 = \frac{16}{25}$$

Solution to Problem #4[4]

The solution to this problem turned out to be rather amazing. When I was playing around with Visio, I could hardly believe the pattern I was seeing. And then it dawned on me—I was seeing the same behavior I found in the "Curve Making Puzzle."[5]

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[2] 4:16 AM • Aug 15, 2020 https://twitter.com/Cshearer41/status/1294548600010297344
[3] 7:16 AM • Aug 31, 2020 https://twitter.com/Cshearer41/status/1300392090397016066
[4] 9:08 AM • Apr 28, 2020 https://twitter.com/Cshearer41/status/1255121674455994368
[5] http://josmfs.net/2020/11/07/curve-making-puzzle/

As shown in Figure 5, as the yellow square pivots about its corner attached to the upper right corner of the blue square and the diagonally opposite corner of the yellow square slides along the orange square, the lower corner of the yellow square slides along the horizontal line bisecting the blue square in half.

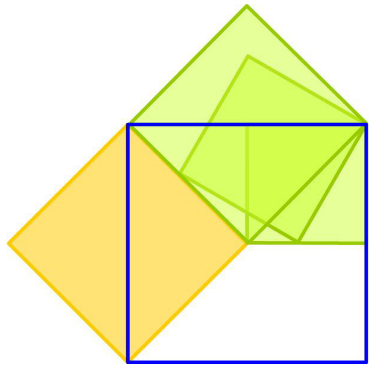

Figure 5

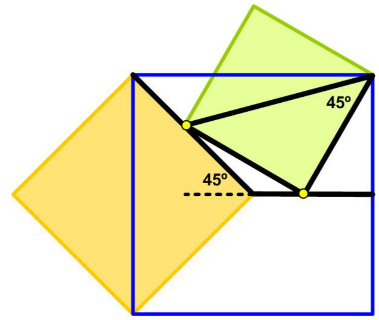

Figure 6

The reason for this is shown in Figure 6 and is the same as in the "Curve Making Puzzle," namely, a series of similar triangles are pivoting about the fixed point. As the endpoint on the longer side of the triangle moves along the edge of the orange square, the endpoint on the shorter triangle side describes a similar curve (a straight line) at the same angle that separates the two sides of the triangle (45°).

And so we arrive at the solution to the problem (Figure 7). Since the left upper corner of the small blue square is attached to the lower right corner of the yellow square along the line bisecting the big blue square, the diagonal of the small blue square must be the same length as the side of the yellow square. Since the area of the small blue square is 2, its side is √2 and diagonal is 2. Therefore the side of the yellow square is 2 and its area is 4.

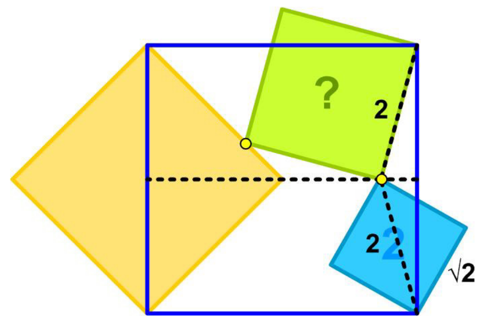

Figure 7

The comments at Catriona's Twitter site for the puzzle are interesting. She challenges a number of solutions that did not prove the lower right corner of the yellow square lay on the bisecting horizontal line.

Solution to Problem #5[6]

This was another nifty problem, which I attacked first using the Polya principle that the size of the circles had not been specified. So the problem must yield the same angle for any pair of circles. I chose the simplest, namely, equal circles (Figure 8). The edges of the square and added diagonal were tangents to the circles and therefore the same length. The resulting isosceles triangles meant that $2\beta = 90° - \alpha$. But the bisecting diagonal meant that $\alpha = 45°$. Therefore the desired angle for the problem was $2\beta = 135°$.

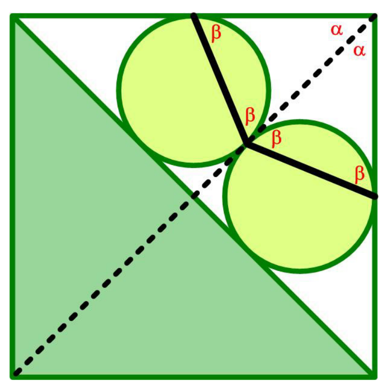

Figure 8

Nevertheless, I took the original statement of the problem to imply one should really show that the size of the circles did not matter, so solved that problem as well.

[6] 6:33 AM • Oct 31, 2020 https://twitter.com/Cshearer41/status/1322486941523300352

Figure 9 shows my approach. The tangent to the two circles (originally the diagonal of the square) was extended until it met the right edge of the square extended. The same argument about equal tangents to the circles obtains, so we have again isosceles triangles. The base angles satisfy

$$\beta = (180° - (90° - \alpha))/2 = 45° + \alpha/2$$

$$\gamma = (180° - \alpha)/2 = 90° - \alpha/2$$

So adding the two to get the desired angle of the problem yields:

$$\gamma + \beta = (90° - \alpha/2) + (45° + \alpha/2) = 135°$$

At Catriona's site there were numerous other solutions proposed involving other properties of the figure.

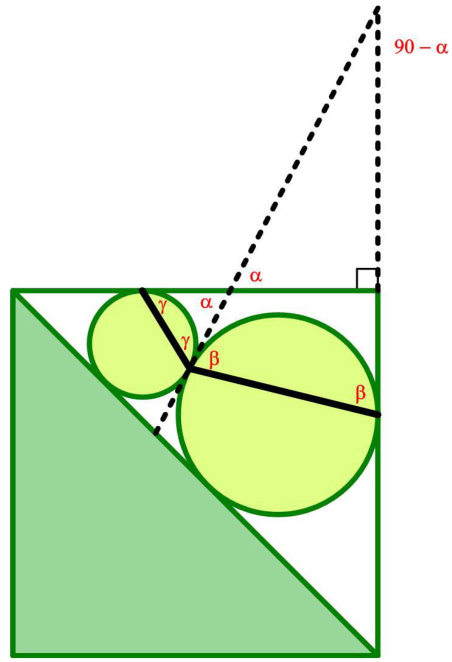

Figure 9

Solution to Problem #6[7]

We end with a fairly simple problem. We assume from the problem statement figure that the blue sides of the angle intersect the circles at their tangent points (not including the vertex of the angle). Then the (dotted) lines joining the centers of the three circles also pass through the three mutual tangent points of the circles as shown. Adding the lengths of the radii (I assumed wlog that the radii were of lengths 1, 2, and 3 rather than the diameters) we have a 3-4-5 triangle, which is a right triangle. Therefore the central angle of the smallest circle is 90°, which implies the desired, inscribed, angle is half that or 45°.

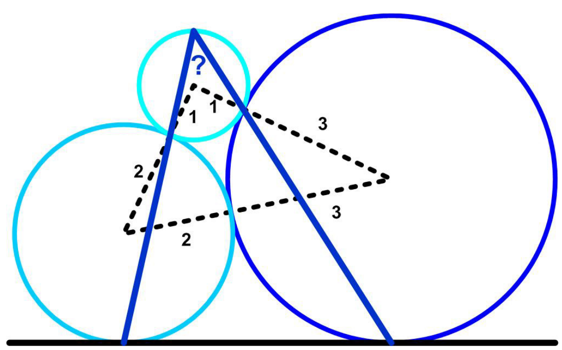

Figure 10

All the commenters at Catriona's site seemed to provide essentially the same solution.

[7] 6:21 AM • Apr 7, 2020 https://twitter.com/Cshearer41/status/1247469594165551105