Autumn Sum

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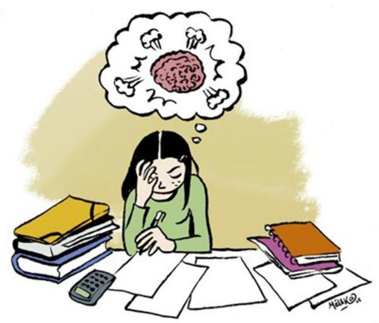

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Here is another problem from the 2020 Math Calendar ([1]).

$$\left(\sum_{n=1}^{\infty} \frac{n}{4^{n-3}}\right) - \frac{4}{9} = ?$$

As a hint, recall that all the answers are integer days of the month. And the solution employs a technique familiar to these pages.

Solution

It is easier to see perhaps if we write out the infinite series. Let S be the sum. Then

$$S = \frac{1}{4^{-2}} + \frac{2}{4^{-1}} + \frac{3}{4^{0}} + \frac{4}{4^{1}} + \frac{5}{4^{2}} + \ldots + \frac{n}{4^{n-3}} + \ldots$$

Factoring out $^1/_4{}^{-2}$,

$$S = \frac{1}{4^{-2}}\left(1 + \frac{2}{4} + \frac{3}{4^2} + \frac{4}{4^3} + \ldots + \frac{n}{4^{n-1}} + \ldots\right) \tag{1}$$

Now for the tried-and-true power series approach with the old geometric series workhorse:

$$G(x) = 1 + x + x^2 + \ldots + x^n + \ldots = \frac{1}{1-x}$$

Taking the derivative,

$$G'(x) = 1 + 2x + 3x^2 + \ldots + nx^{n-1} + \ldots = \frac{1}{(1-x)^2}$$

Evaluating at $x = ¼$,

$$G'\left(\tfrac{1}{4}\right) = 1 + \frac{2}{4} + \frac{3}{4^2} + \frac{4}{4^3} + \ldots + \frac{n}{4^{n-1}} + \ldots = \frac{4^2}{3^2} = \frac{16}{9}$$

So

$$S = \frac{1}{4^{-2}} G'\left(\tfrac{1}{4}\right) = \frac{16^2}{9}$$

and

$$\boxed{S - \frac{4}{9} = \frac{16^2 - 4}{9} = \frac{4(64-1)}{9} = 28}$$

Historical Note

The series in equation (1) in the form

$$\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\ldots=\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\ldots+\frac{n}{2^n}+\ldots=\frac{1}{2}\left(1+\frac{2}{2}+\frac{3}{2^2}+\ldots+\frac{n}{2^{n-1}}+\ldots\right)=\frac{1}{2}G'(\tfrac{1}{2})=\frac{4}{2}=2$$

shows up in medieval mathematics, considered by Richard Swineshead (Suiseth), aka the Calculator (fl. c.1340-1354),[1] only in verbal form. Swineshead found the sum without using power series, but still using the geometric series. Judith Grabiner[2] presents his solution in more modern notation:

The scholars at Merton College explored other mathematically interesting ways that a form could vary. For instance, suppose a form has the intensity of 1 during the first half of a period of time, 2 the first half of the time remaining, 3 the first half of the time yet remaining, and so on ad infinitum. Then its total intensity would be given by, in our notation, the sum of terms of the form $k/2^k$ from $k = 1$ to infinity. That is:

$$1/2 + 2/4 + 3/8 + 4/16 + 5/32 + \cdots .$$

What is this total? I gave this problem to a mathematical audience, and a distinguished topologist quickly reinvented Swineshead's method. Swinsehead did it in words;[3] here is the solution in the fraction and sum notation we use now:

$$1/2 + 2/4 + 3/8 + 4/16 + 5/32 + \cdots\cdots\cdots =$$

$$1/2 + 1/4 + 1/8 + 1/16 + 1/32 + \cdots = 1 \qquad \text{[geometric series]}$$
$$+ 1/4 + 1/8 + 1/16 + 1/32 + \cdots = 1/2$$
$$+ 1/8 + 1/16 + 1/32 + \cdots = 1/4$$
$$+ 1/16 + 1/32 + \cdots = 1/8$$
$$+ 1/32 + \cdots = 1/16$$
$$\ldots$$

Adding these separate sums vertically gives the total = **2**.

Oresme gave the same problem, drew a diagram for it, and did the sum by an ingenious geometrical argument. Then he solved a similar geometric problem using the same time intervals, only this time the motion was continuous, uniform in the odd-numbered intervals, and uniformly accelerated in the even-numbered ones.

References

[1] Rapoport, Rebecca and Dean Chung, *Mathematics 2020: Your Daily epsilon of Math*, Point Rock, Quarto Publishing Group, New York, 2020. October

[2] Boyer, Carl B., *The History of Calculus and Its Conceptual Development (The Concepts of the Calculus)*, Dover Publications, Inc., New York, 1949

[3] Grabiner, Judith V., "Why Should Historical Truth Matter to Mathematicians? Dispelling Myths while Promoting Maths," Chapter 9 in *A Historian Looks Back: The Calculus as Algebra and Selected Writings*, Mathematical Association of America, 2010. pp. 243-255

1 JOS: [2] p.69
2 JOS: [3] p.248
3 JOS: [2] pp.77-78