Sum Of Squares Puzzle

26 October 2019

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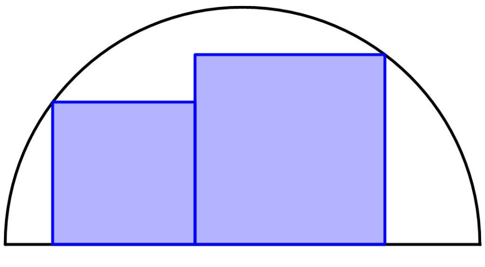


Yet another interesting problem from Presh Talwalkar.[1]

Two side-by-side squares are inscribed in a semicircle. If the semicircle has a radius of 10, can you solve for the total area of the two squares? If no, demonstrate why not. If yes, calculate the answer.

This puzzle shares the characteristics of all good problems where the information provided seems insufficient.

My Solution

I have to admit my approach was not as slick as Talwalkar's (which I show below), but it was another example of seeing possible relationships via the Viseo application and then having to prove them. The first thing I did was assign coordinates to the vertex of the larger square intersecting the semicircle (Figure 1) and then draw the radius from the center of the semicircle to the intersection point. I then drew a second radius to the vertex of the smaller square intersecting the semicircle (Figure 1). It "looked" like the two radii made a 90° angle. That reminded me of the idea of rotating the xy-rectangle at (x, y) 90° as also shown in Figure 2. This also showed that the horizontal distance between the original point and the rotated point was x + y.

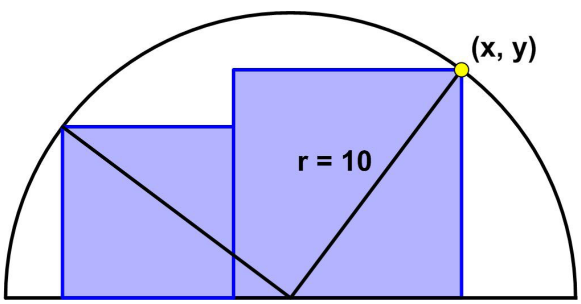


Figure 1

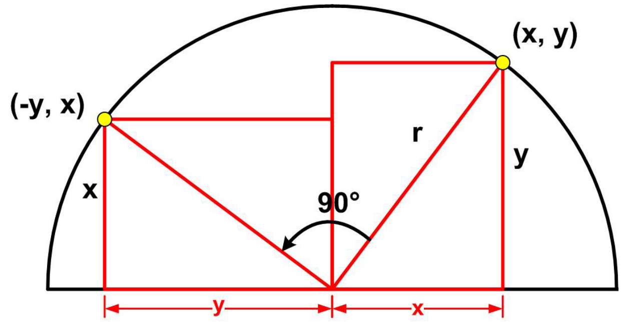


Figure 2

I then reinserted the squares back into the figure (Figure 3). Again, it "looked" like the smaller square fit exactly inside the rotated rectangle. But that needed to be proved.

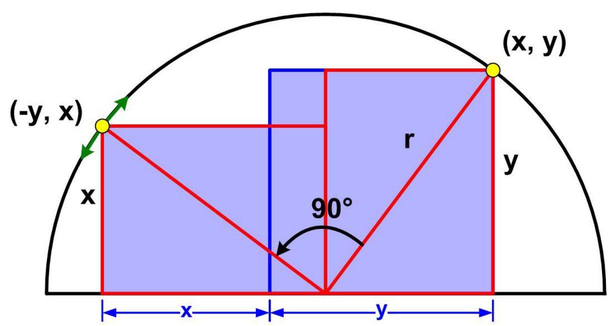


Figure 3

By definition the y-coordinate represented the edge of the large square. Moreover, since the horizontal distance between the two points was x + y, the figure did show a smaller square with edges x. But was that the original second square? That is, could there be another square fitting between the large square and the semicircle. As Figure 3 shows, any change of the second, rotated point along the semicircle will shorten one edge

[1] https://mindyourdecisions.com/blog/2019/10/25/sum-of-squares-puzzle/

and lengthen the adjacent edge of the square to form a rectangle which is not a square. Therefore, given the large square (fixed y), there is only one possible smaller square that can be inscribed next to it in the semicircle. Thus our original smaller square is the one with edges x. That means the sum of the areas of the two squares is

$$x^2 + y^2 = r^2 = 100.$$

Comment. From the point of view of a puzzle, we can apply Polya's idea that if something isn't specified, choose a special case that is simple. In this problem, the size of the large square (or equivalently, the location of the vertex (x, y)) is not specified. Therefore, choose a size that is simple, namely, choose equal squares. By symmetry this places a lower vertex of each square at the center of the semicircle. Therefore the diagonal of each square is the radius r = 10, and so the edge of each square is 5√2. This makes the area of each square be 25·2 = 50, and thus the area of both squares 100.

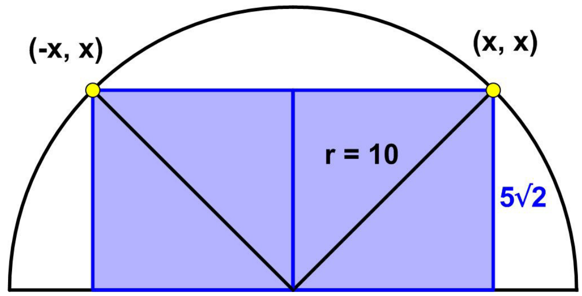


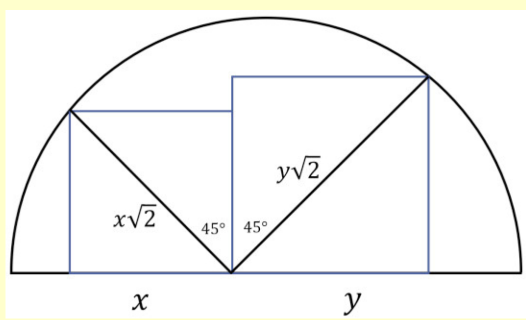
Figure 4 Equal Squares

Talwalkar Solution

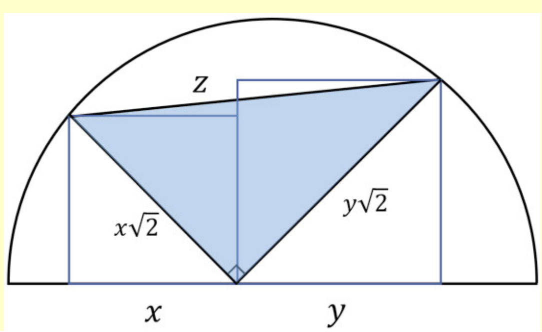
Let the squares have sides x and y. The total area of the squares is then:

$$\text{Area} = x^2 + y^2$$

Draw diagonals in the squares meeting at their common corner. The diagonals bisect the corner angles, so each diagonal creates a 45 degree angle. And together the two diagonals form a right angle—meaning the arc has a measure of 90 degrees. Each diagonal also has a length of √2 times the side length of the square.



Draw a chord of the circle connecting the other endpoints of the diagonals of the squares, and label the side z.

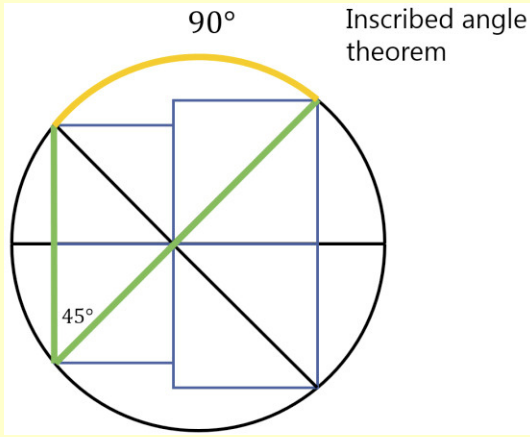


The sides $x\sqrt{2}$, $y\sqrt{2}$, and z form a right triangle. Thus by the Gougu theorem:[2]

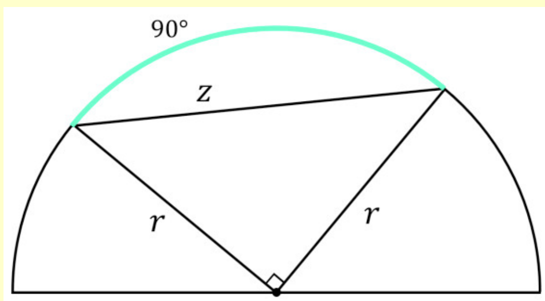
$$(x\sqrt{2})^2 + (y\sqrt{2})^2 = z^2$$

$$2x^2 + 2y^2 = z^2$$

Let's also find the arc subtended by the chord with length z. Reflect the semicircle and squares across the diameter. Consider the inscribed angle formed by the two sides of the small square and the diagonals across both squares. This is a 45 degree angle since the diagonal bisects the corner angle in a square. By the inscribed angle theorem, the inscribed angle subtends a circular arc of double the measure, 2(45) = 90 degrees.



From the center of the circle, draw radii to the endpoints of the chord with length z. Since the chord subtends the 90 degree angle, the central angle must also be a 90 degree right angle.



Thus the sides r, r, and z form a right triangle, and by the Gougu theorem we have:

$$r^2 + r^2 = z^2$$

$$2r^2 = z^2$$

We have two equations equal to z^2, so we set them equal to each other.

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[2] JOS: Talwalkar explains in his posting why he is calling the Pythagorean Theorem the Gougu theorem—because Pythagoras neither stated nor proved his eponymous theorem. Other early cultures were aware of the relationship (Egypt, Babylonia), but according to Talwalkar's sources Gougu in China was the first to prove it. The sentiment to try to assign original discovery to the correct person is basically laudable, but in general I find the effort somewhat futile and a distraction. In my day Cauchy's Inequality became the Cauchy-Schwartz-Bunyakovsky Inequality in order to give more accurate credit. But we know that Jordan did not prove Jordan's Curve Theorem and other names are attached to results whose persons did not prove (Fermat's Last Theorem!). Many results in mathematics are discovered in parallel and presented in isolated sources that are only found later. It is fair to try to set the record straight, but it can get a bit arbitrary and contentious if carried too far.

$$z^2 = 2x^2 + 2y^2 = 2r^2$$

$$x^2 + y^2 = r^2$$

Recalling that the total area is $x^2 + y^2$, we can get the answer by substituting $r = 10$.

$$\text{Area} = x^2 + y^2 = r^2$$

$$\text{Area} = 10^2 = 100$$

And the problem works out very neatly in the end! The area is the same regardless of how the two side by side squares are inscribed.

Sources

Math StackExchange

https://math.stackexchange.com/questions/3365745/two-side-by-side-squares-are-inscribed-in-a-semicircle-the-diameter-of-the-semi

Nèstor Abad solution to puzzle (posted by @CShearer41)

https://twitter.com/nabadvin/status/1165195248856944641

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