# Serious Series

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The following problem comes from a 1961 exam set collected by Ed Barbeau of the University of Toronto. The discontinued exams (by 2003) were for 5th year Ontario high school students seeking entrance and scholarships for the second year at a university ([1]).

If $s_n$ denotes the sum of the first $n$ natural numbers, find the sum of the infinite series

$$\frac{s_1}{1}+\frac{s_2}{2}+\frac{s_3}{4}+\frac{s_4}{8}+\ldots.$$

Unfortunately, the "Grade XIII" exam problem sets were not provided with answers, so I have no confirmation for my result. There may be a cunning way to manipulate the series to get a solution, but I could not see it off-hand. So I employed my tried and true power series approach to get my answer. It turned out to be power series manipulations on steroids, so there must be a simpler solution that does not use calculus. I assume the exams were timed exams, so I am not sure how a harried student could come up with a quick solution. I would appreciate any insights into this.

## My Solution

The first step was to convert $s_n$ into its usual sum:

$$s_n = 1+2+3+\ldots+n = \frac{n(n+1)}{2}$$

Then the desired series becomes

$$\sum_{n=1}^{\infty}\frac{s_n}{2^{n-1}} = \sum_{n=1}^{\infty}\frac{n(n+1)}{2^n}$$

Now for the usual trick of converting this to a power series

$$f(x) = \sum_{n=1}^{\infty} n(n+1)x^n$$

where we want to find $f(½)$.

The $n + 1$ factor reminded me of integration, so I integrated the power series term-by-term:

$$F(x) = \int_0^x f(t)dt = \sum_{n=1}^{\infty}\int_0^x n(n+1)t^n dt = \sum_{n=1}^{\infty} n(n+1)\left.\frac{t^{n+1}}{n+1}\right]_0^x = \sum_{n=1}^{\infty} nx^{n+1} = x^2\sum_{n=1}^{\infty} nx^{n-1} = x^2 g'(x)$$

where $g'(x)$ is the derivative of our old friend the geometric series

$$g(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

So

$$f(x) = F'(x) = 2xg'(x) + x^2 g''(x)$$

where

$$g'(x) = \frac{1}{(1-x)^2} \quad \text{and} \quad g''(x) = \frac{2}{(1-x)^3}$$

so that

$$f(x) = \frac{2x}{(1-x)^2} + \frac{2x^2}{(1-x)^3} = \frac{2x}{(1-x)^3}$$

Therefore,

$$\boxed{f\left(\tfrac{1}{2}\right) = 8 = \sum_{n=1}^{\infty} \frac{s_n}{2^{n-1}}}$$

Amazing!

## References


[1] "Grade XIII Problems", Annual Examinations, Department of Education, Ontario, 1961. To be taken only by candidates writing for certain University Scholarships involving Mathematics. (http://www.math.utoronto.ca/barbeau/ontprob1961.pdf, retrieved 12/15/2019)