Circle Projection Problem

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This is a nice Brainteaser from the *Quantum* math magazine ([1]).

Line segment MN is the projection of a circle inscribed in a right triangle ABC onto its hypotenuse AB. Prove that angle MCN is 45°.

My Solution

First, I dropped a perpendicular from the vertex of the right triangle and labeled the angles the diagonal lines made with the vertical lines α and β, as shown in Figure 1. The angles are equal as shown in the figure via the transverse line cutting parallel lines result.

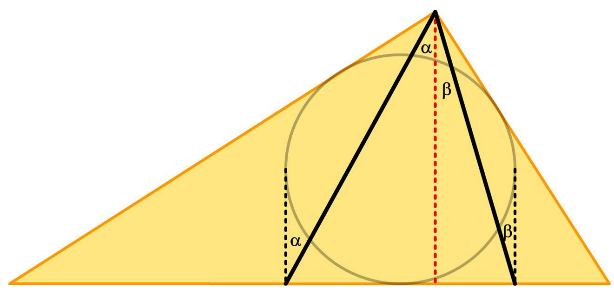

Figure 1 Solution Step 1

As Figure 2 shows, the tangents to the inscribed circle are equal since the are made up of the blue equal tangents and the red sides of the equal squares. That means they form an isosceles triangle with the left slanted line, and so the base angles α are equal. Figure 3 shows the same argument for the right slanted line and angle β. Thus

$$2\,\alpha + 2\,\beta = 90°$$

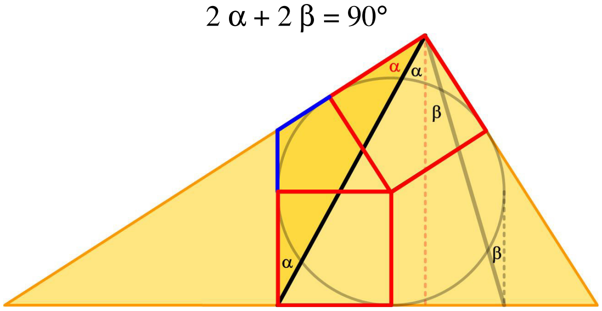

Figure 2 Solution Step 2

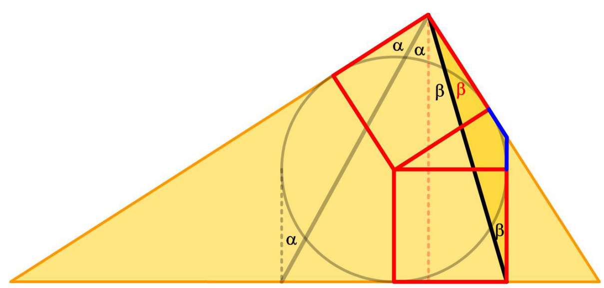

Figure 3 Solution Step 3

and so $\alpha + \beta = 45°$, which is what we wanted to show.

Quantum Solution

Let O be the center of the circle inscribed in triangle ABC (Figure 4). We note that triangles POM, QOC are congruent,[1] so that OM = OC. Similarly ON = OC. Hence O is also the center of the circle circumscribed about the triangle MCN. It is not hard to see that triangles POM, RON are congruent isosceles right triangles, and it follows that

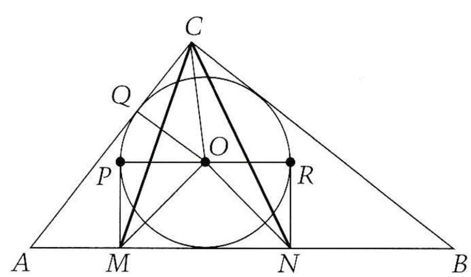

Figure 4 *Quantum* Solution

[1] JOS: This is really the argument I gave in Figure 2 involving the red squares.

∠MON = 90°. Therefore, ∠MCN = 45°, since an inscribed angle is half as large as a central angle with the same arc.

Comment

It took me a while to find the solution I gave. Again I used Visio and had drawn all sorts of lines that revealed interesting relationships. I latched onto the isosceles triangle idea, but then had to find a proof. Slowly the pattern for a proof revealed itself through the many possibilities, yielding the final satisfying result.

I compared this feeling with how I felt trying to solve numerical problems that involve a fair amount of computation. Since invariably I make arithmetic mistakes, trying to correct the result is like trying to balance your checkbook—a thoroughly unpleasant activity with little satisfaction when it is done, since the correction is mindless and does not involve the kind of insight that comes with suddenly seeing the pattern of a geometric proof. So I guess that is why I gravitate to geometry problems instead of arithmetic problems. This is a sad admission, since traditionally mathematics has involved sometimes hairy computations, as witnessed by the eminent 18^{th} and 19^{th} century mathematicians. And then, of course, the truly great masters of computation are the physicists, and especially with their prowess at making estimations. Perhaps that is one of the reasons I fled physics for math.

References

[1] "The Right Approach" B292 "Brainteasers" *Quantum* Vol.10, No.5, National Science Teachers Assoc., Springer-Verlag, May-Jun 2000. p.3