The Triangle of Abū'l-Wafā'

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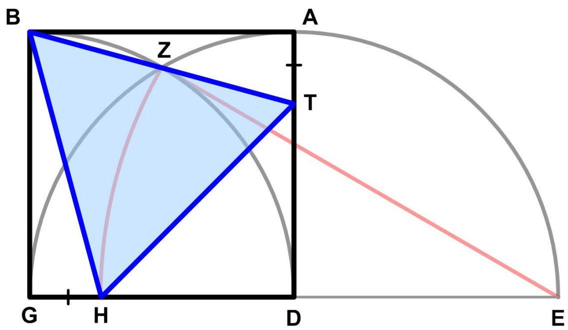

I found an interesting geometric statement in a paper of Glen Van Brummelen ([1]) cited in the online MAA January issue of *Convergence*:

> For instance, Abū'l-Wafā'[1] describes how to embed an equilateral triangle in a square, as follows: extend the base GD by an equal distance to E. Draw a quarter circle with centre G and radius GB; draw a half circle with centre D and radius DE. The two arcs cross at Z. Then draw an arc with centre E and radius EZ downward, to H. If you draw AT = GH and connect B, H, and T, you will have formed the equilateral triangle.

So the challenge is to prove this statement regarding yet another fascinating appearance of an equilateral triangle.

Solution

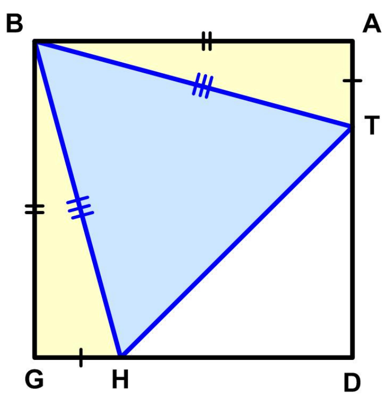

Figure 1

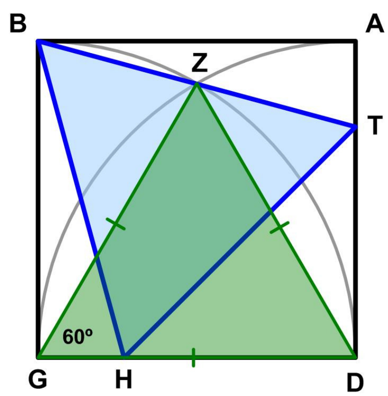

Figure 2

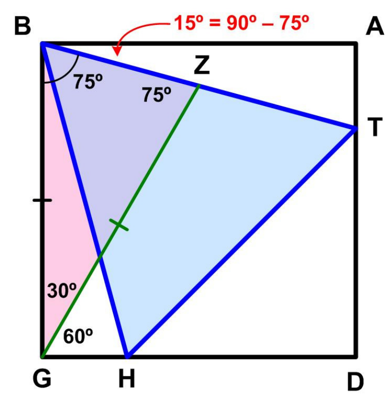

Figure 3

The first thing to notice (Figure 1) is the yellow triangles are right triangles, and since the short sides are given equal and the long sides are the equal sides of the square, then the hypotenuses must be equal, so that the yellow triangles are congruent. Therefore the blue triangle is isosceles. If we can show the vertex angle of the blue triangle at the corner of the square is 60º, then the other two angles must also be 60º, making the triangle equilateral.

Join two of the corners of the square to the intersection point Z to produce the green triangle of

[1] *Wikipedia*: (940 – 998) a Persian mathematician and astronomer who worked in Baghdad. He made important innovations in spherical trigonometry, and his work on arithmetics for businessmen contains the first instance of using negative numbers in a medieval Islamic text. He is also credited with compiling the tables of sines and tangents at 15 ' intervals. He also introduced the secant and cosecant functions, as well studied the interrelations between the six trigonometric lines associated with an arc. His *Almagest* was widely read by medieval Arabic astronomers in the centuries after his death. He is known to have written several other books that have not survived.

Figure 2 That triangle must be equilateral since two of the legs are radii of the same sized circles generated from the edge of the square, which is the third side of the green triangle. So its lower left vertex angle must be 60º and its complementary angle 30º. The red triangle is also isosceles, since the two edges of the 30º angle are equal to the sides of the square (Figure 3). That means the base angles are 75º each. The complement of 75º is 15º.

But since 15º is a vertex angle of one of the congruent yellow triangles, so must the other corresponding angle be 15º, or 30º total (Figure 4). That means the vertex angle of the blue triangle must be 90º – 30º = 60º, which is what we wanted to show. Therefore, the blue triangle is equilateral.

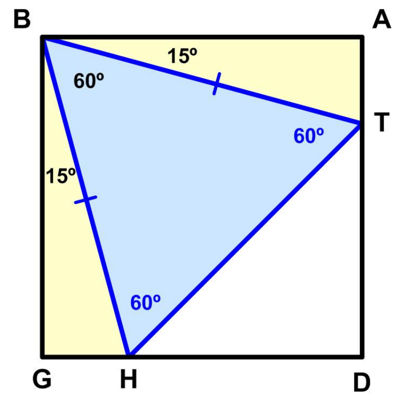

Figure 4

References

[1] Van Brummelen, Glen, "Why History of Mathematics?" *Vector*, Vol. 60, No. 2 (Fall 2019), pp.19-23. (https://www.maa.org/sites/default/files/images/upload_library/46/NCTM/Why_History_of_Math_VECTOR_Fall2019_pp19-23.pdf) cited in MAA *Convergence* (January 2020) (https://www.maa.org/press/periodicals/convergence/why-history-of-mathematics)