Quadrangle in Parallelogram

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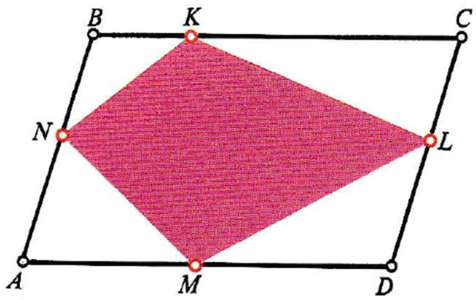


Here is another problem from the *Quantum* magazine ([1]), only this time from the "Challenges" section (these are expected to be a bit more difficult than the Brainteasers).

A quadrangle is inscribed in a parallelogram whose area is twice that of the quadrangle. Prove that at least one of the quadrangle's diagonals is parallel to one of the parallelogram's sides. (E. Sallinen)

My Solution

I thought I would begin with a quadrilateral that did not ostensibly have a diagonal parallel to a side of the parallelogram (Figure 1).

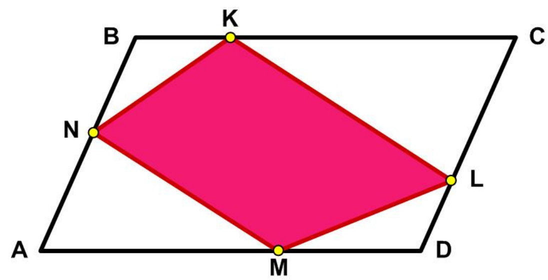


Figure 1 General Quad in Parallelogram

It would be easier to deal with a rectangle instead of a parallelogram, so the first thing is to show that would not change the problem.

Shear Invariance. This is an idea I have broached before with triangles. If we shift all the horizontal lines of Figure 1 to the left so that ABCD becomes a rectangle, the quadrangle (and rectangle ABCD) will have the same area as before. Moreover, all lines parallel to the vertical edges of the parallelogram will be parallel to the vertical edges of the rectangle. And given the horizontal shear of the figure all the horizontal lines remain parallel to the horizontal edges of the figure.

This can be seen in Figure 2 where we have shown one of the triangles we could use to decompose the area of the quadrangle into a tiling of triangles. Since the horizontal lines are moved horizontally and in parallel, the altitude h of the triangle remains unchanged between the parallelogram and rectangle. Similarly, the horizontal shear motion is without compression or dilation, so the base b of the triangle is also unchanged. Therefore the area of the triangle is unchanged.

So if we can prove the case for a quadrangle in a rectangle, then that will suffice for any parallelogram.

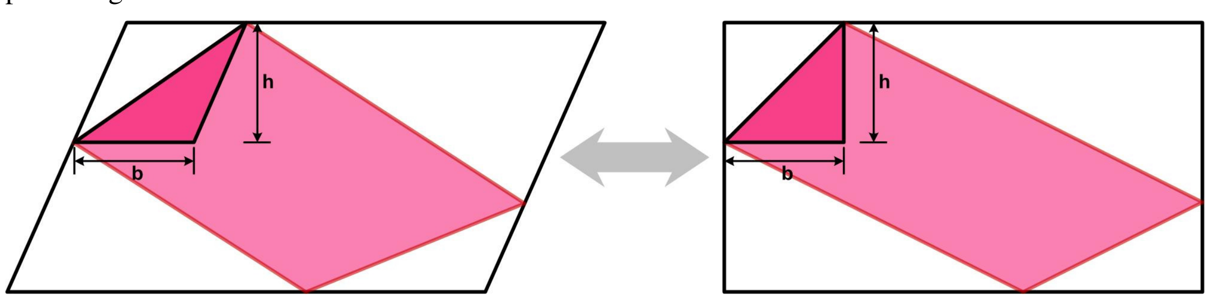
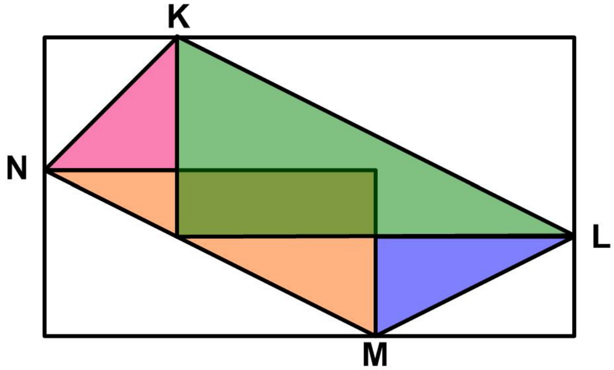
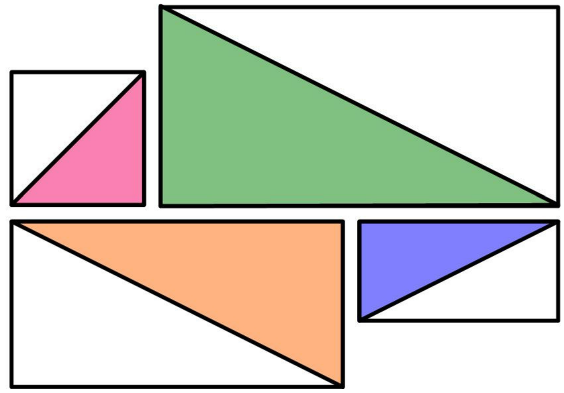


Figure 2 Shear Invariance

Now decompose the inscribed quadrangle into four (possibly overlapping) right triangles where the four hypotenuses are the four sides of the quadrangle (Figure 3).



(overlapping view)



(exploded view)

Figure 3 Decomposition of Quadrangle into Four Right Triangles

The exploded view of Figure 3 shows that each colored triangle is half the area of its circumscribed rectangle. Since the triangles overlap in the decomposition on the left, the colored area is less than half the total area of the rectangle (the overlap area only gets counted once). In order to have the colored area (the area of the quadrilateral) be half the rectangle, the triangles must not overlap.

Therefore the overlap area must be zero. That means that either the vertical sides of the overlap rectangle are zero length., so NL is a single straight line (parallel to the horizontal edge of the rectangle) and a diagonal of the quadrilateral, or the horizontal sides of the overlap rectangle are zero length, so KM is a single straight line (parallel to the vertical edge of the rectangle) and a diagonal of the quadrilateral.

I rather like this argument, since it makes it visually clear how the statement of the claim corresponds to the half area.

Quantum Solution

To expose the main idea of the proof let's consider a triangle XYZ with a fixed base YZ and vertex X moving along a line l (Figure 4). It's quite obvious that the area of XYZ is constant if l is parallel to YZ, otherwise, it varies monotonously as long as X doesn't cross YZ. (Actually, the area is proportional to the distance from X to YZ.)

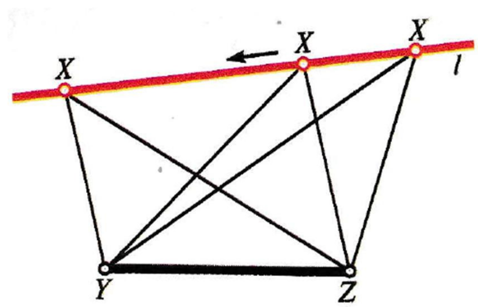


Figure 4 ***Quantum*** **Solution Step 1**

Now let's denote the given quadrangles as in Figure 5. If a diagonal of the inscribed quadrangle, say KM, is

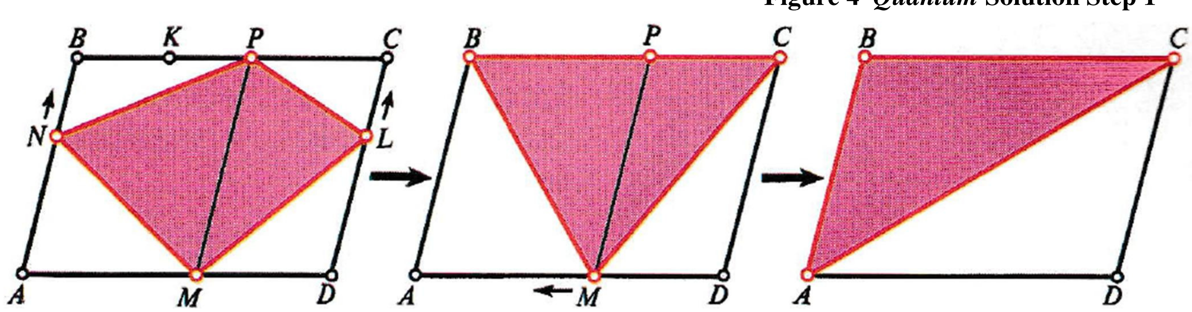


Figure 5 ***Quantum*** **Solution Step 2**

parallel to a side of the parallelogram (AB or CD), we're done. Otherwise we mark the point P on BC such that PM is parallel to AB. Using our moving method it's easy to transform PLMN into the triangle ABC, which is just half of ABCD, so its area remains unchanged. Thus the quadrangles KLMN and PLMN have the same area, equal to half the area of ABCD.

Subtracting the triangle LMN from both of these quadrangles, we get two triangles, LNK and LNP (Figure 6), with the common base LN and equal areas. From this it follows that KP (or the side BC) is parallel to the diagonal LN. (V. Dubrovsky)

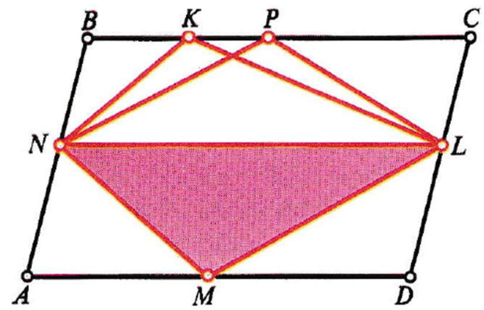


Figure 6 Quantum Solution Step 3

References

[1] "Challenges" *Quantum Magazine*, National Science Teachers Assoc., Springer-Verlag, Vol. 1 No.1, Sep-Oct 1990, p.23 M13