The Chord Progression Puzzle

24 October 2019

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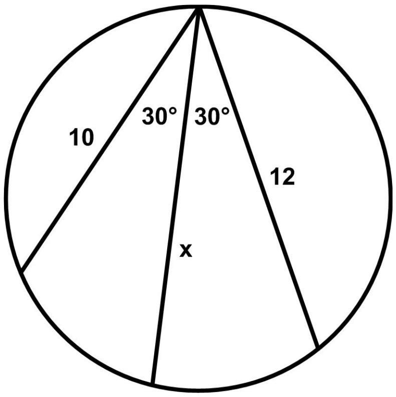

This is another challenging puzzle from Presh Talwalkar[1] that seems difficult to know where to start.

Given the figure shown at left, what is the value of x?

My Solution

My first thought was to find right triangles, since the sine and cosine of 30° are well-known (1/2 and √3/2, respectively). So I dropped perpendiculars from the intersection points of the two measured chords onto the chord of unknown length *x* (Figure 1). I then computed the lengths of the corresponding legs using the trigonometric values (Figure 1). Next I connected the extremities of the chords, drew radial lines from the center, and filled in the appropriate values (Figure 2). Since the inscribed angles in the circle were each 30°, their corresponding central angles would be twice that or 60°. Since each triangle was isosceles with a 60° vertex, the other two vertices had to be 60°, thus making the triangle equilateral. Hence, the lines connecting the extremities of the chords had to be equal to the radius of the circle *r*.

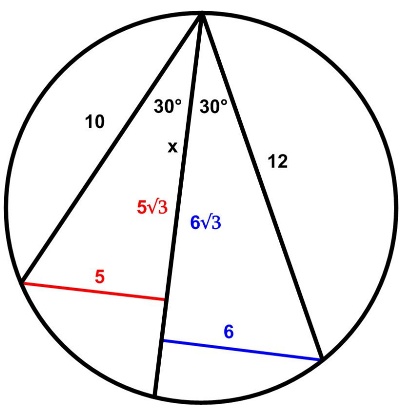

Figure 1 Step 1

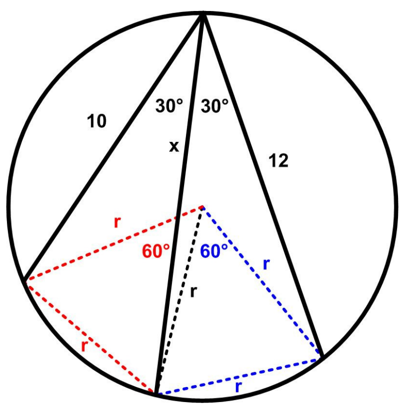

Figure 2 Step 2

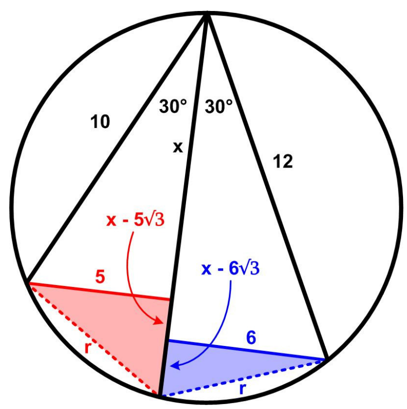

Figure 3 Step 3

The final step was to apply the Pythagorean Theorem to the shaded triangles in Figure 3:

$$r^2 = 5^2 + (x - 5\sqrt{3})^2 = 5^2 + x^2 - 10\sqrt{3}x + 5^2\cdot 3 = x^2 - 10\sqrt{3} + 5^2\cdot 4 \tag{1}$$

$$r^2 = 6^2 + (x - 6\sqrt{3})^2 = 6^2 + x^2 - 12\sqrt{3}x + 6^2\cdot 3 = x^2 - 12\sqrt{3} + 6^2\cdot 4 \tag{2}$$

Subtracting equation (1) from equation (2) yields

[1] https://mindyourdecisions.com/blog/2019/10/22/the-chord-progression-puzzle/

$$4 \cdot (6^2 - 5^2) = 2\sqrt{3}\, x$$

or $$x = 22/\sqrt{3}$$

Talwalkar then proposed solving the general problem where the lengths of the two known chords and the two equal angles are arbitrary (Figure 4).

All we need do is substitute a and b in equations (1) and (2) wherever we used 10 and 12, respectively, and cos θ and sin θ wherever we used cos 30° and sin 30°, respectively. This gives

$$r^2 = (a \sin\theta)^2 + (x - a\cos\theta)^2 = a^2 + x^2 - 2ax\cos\theta$$

$$r^2 = (b \sin\theta)^2 + (x - b\cos\theta)^2 = b^2 + x^2 - 2bx\cos\theta$$

So subtracting the first equation from the second yields

$$2(b - a)x\cos\theta = b^2 - a^2$$

or $$x = (a + b)/(2\cos\theta)$$

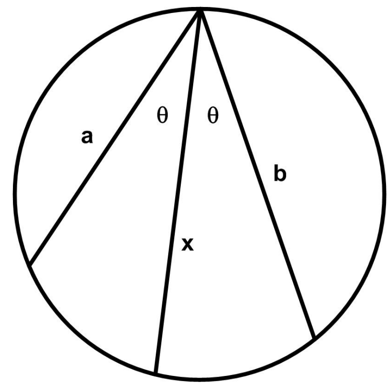

Figure 4 General Problem

The above derivation assumes $a \neq b$. If $a = b$, then arguing by symmetry, the x chord must be a diameter. This means the a-r-x triangle is a right triangle, and so $x = a/\cos\theta$ which still satisfies the derived formula.

Talwalkar Solution

Talwalkar's solution is a bit different. Instead of using the (simpler?) Pythagorean Theorem, he employs the law of cosines, which is not that difficult, but it is harder to remember since it is not used as much as the Pythagorean Theorem. I prefer to use more basic, easily remembered results wherever I can.

First, draw chords to form a quadrilateral as shown in Figure 5. We will prove the two chords at the bottom have equal length $y = z$.

Two inscribed angles for the same arc of a circle have the same measure by the inscribed angle theorem. Thus, we can deduce the following two angles are 30 degrees (Figure 6, Figure 7).

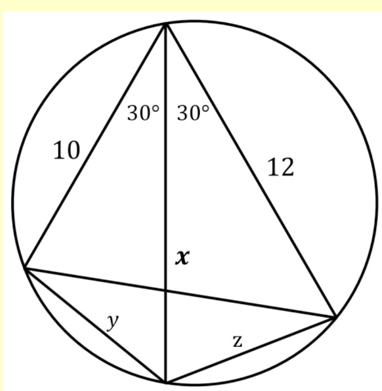

Figure 5

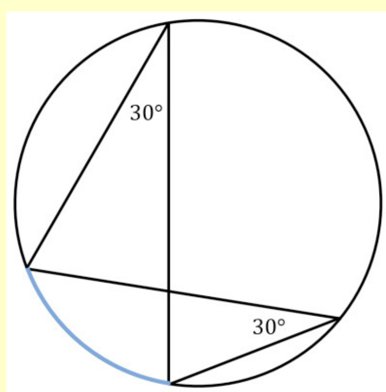

Figure 6

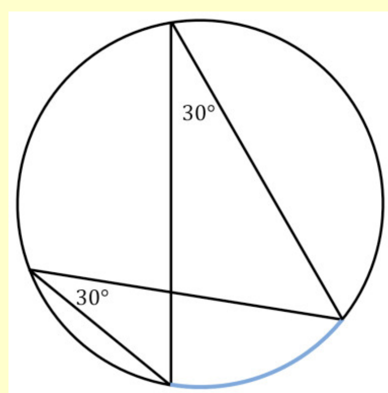

Figure 7

Now consider the triangle at the bottom of the circle that has two angles of 30 degrees (Figure 8). The sides opposite those angles must have equal length y. Thus we have proved $y = z$ (Figure 9).

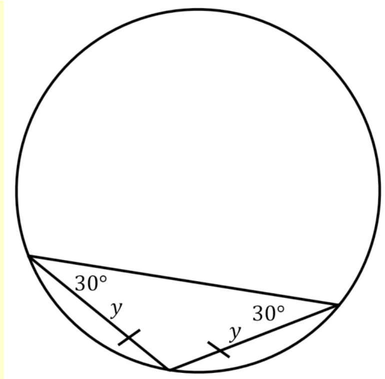

Figure 8

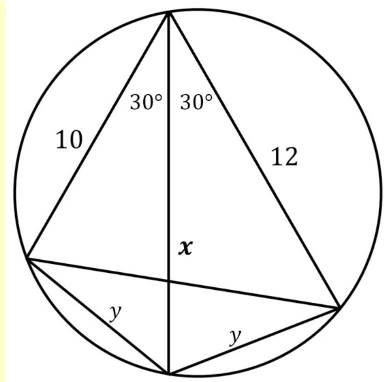

Figure 9

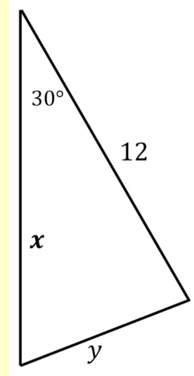

Figure 10

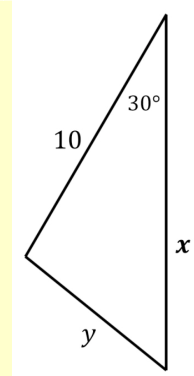

Figure 11

Now consider the triangle with sides y, 12, and x (Figure 10). By Al-Kashi's[2] law of cosines we have:

$$y^2 = 12^2 + x^2 - 2(12)x \cos 30°$$

Now consider the triangle with sides y, 10, and x (Figure 11). Again by Al-Kashi's law of cosines we have:

$$y^2 = 10^2 + x^2 - 2(10)x \cos 30°$$

As both equations are equal to y^2, we can set them equal to each other.

$$12^2 + x^2 - 2(12)x \cos 30° = 10^2 + x^2 - 2(12)x \cos 30°$$

The x^2 terms will cancel out, and then it is routine to solve for x.

$$12^2 - 2(12)x \cos 30° = 10^2 - 2(12)x \cos 30°$$

$$12^2 - 10^2 = 2(12)x \cos 30° - 2(12)x \cos 30°$$

$$12^2 - 10^2 = 4x \cos 30°$$

$$x = (12^2 - 10^2)/\cos 30°$$

$$x = 22/\sqrt{3}$$

Another way we could write this answer is:

$$x = 0.5(10 + 12)/\cos 30°$$

General Case

Indeed, if we consider the general case (Figure 12):

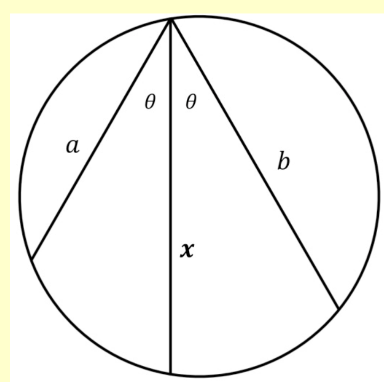

Figure 12

We can use the same method as above to derive the general formula:

$$x = 0.5(a + b)/\cos \theta$$

I think this is a pretty neat formula. It states the middle chord is a kind of "average"–it is the simple average of the two chords multiplied by the adjustment factor of $1/\cos \theta$.

This has probably been discovered thousands of years ago. But

[2] JOS: This is Talwalkar's idiosyncrasy where he tries to assign to well-known theorems names that recent sources suggest were the original discoverers. This is not an unprecedented practice, but it can be confusing to the uninitiated.

I never learned it, so it was a fun "discovery" for me!

Source

Adapted from Reddit HomeworkHelp

https://www.reddit.com/r/HomeworkHelp/comments/d7bh9k/high_school_math_geometry_how_do_i_solve_for_the/