

Rubber Band Ant

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This is a stimulating little problem from the ever-creative James Tanton:¹

An ant is at the east end of an infinite stretchy band, initially 2 ft long. Each day: ant walks 1 ft west on the band. Overnight while sleeping, band stretches to double its length (carrying ant westward as does so). Same routine each day/night. Will ant cover 99% of band's length?

(Ant from clipart-library.com)

My Solution

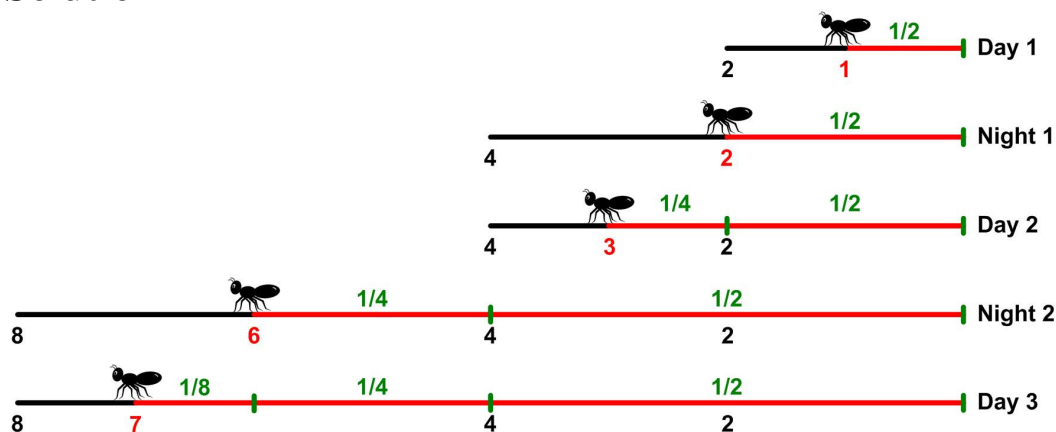


Figure 1 Ant and Rubber Band Positions after Day 3

We illustrate a few of the steps and then the overall pattern will become clear. Figure 1 shows the situation after 3 days. The stretching of the rubber band each night does not change the fractions of progress the ant has already made on the previous days. If each day's new length of the rubber band is L_n , then the fraction of the ant's progress of one foot that day is $1/L_n$, which is $1/2$ the progress of the previous day, since $L_n = 2 L_{n-1} \Rightarrow 1/L_n = 1/2 \cdot 1/L_{n-1}$. So we have that the fractional total distance traveled by the ant is a geometric progression with $r = 1/2$. Let S_n be the total fractional progress the ant has made along the rubber band after day n . Then we have

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{1}{4}$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$$

$$S_n = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \left(\frac{1}{2}\right)^n$$

¹ Tanton, James, *Twitter*, 20 March 2020 (<https://twitter.com/jamestanton/status/1240999581392896000>)

From this we get, for $r = 1/2$,

$$S_n = r \frac{1 - r^n}{1 - r} = 1 - r^n \quad (*)$$

To see if eventually $S_n > .99$, we want

$$S_n > 1 - .01$$

or

$$1 - r^n > 1 - .01$$

or

$$r^n = 1/2^n < 1/100$$

or

$$2^n > 100$$

But $2^6 = 64$ and $2^7 = 128$. So for $n = 7$ (Day 7) the ant's progress has exceeded 99% of the length of the rubber band.

Other Solutions

Two commenters provided answers for Tanton's problem.

Darren Abbruzzese Mar 20 (<https://twitter.com/abbruzzd/status/1241091451305385984>)

After each days walking and overnight growing the band's length has doubled and the ant's position remains 2ft away from the west end. By the end of day 7 the band is 256ft long and the ant is 254ft west, or 99.2% of the distance.

Alternatively, he could say at the end of each day he remained 1 foot away from the west end. In any case, that probably needs a little bit of argument. Actually, it is equivalent to equation (*) where we see that

$$1 - S_n = r^n = 1/2^n = 1/L_n$$

which represents a foot. That is, the ant is always a foot away from the west end at the end of each day.

Andreas Steiger Mar 21 (<https://twitter.com/mittelwertsatz/status/1241268106892378112>)

The relative position stays the same as in the situation where the band doesn't stretch but the ant keeps covering only half of what it did the last day.

Sort of my solution, only he would have to argue a bit about not having to stretch the rubber band. Possibly the "fraction" representation is equivalent to that.

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