Meditation on “Is” in Mathematics Part III – Heliocentrism

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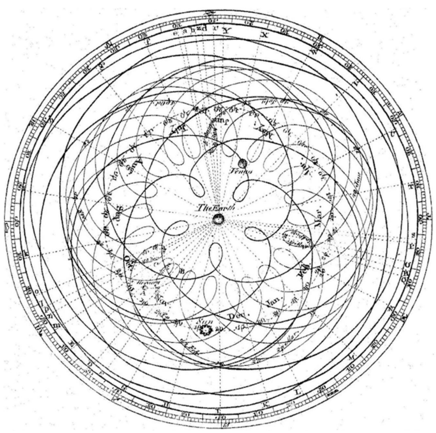

Given the aggravating times, I thought I would vent my frustration by ranting on a somewhat nonsensical topic: “The *fact* that the earth revolves around the sun, rather than the sun around the earth.” This assertion is often used to separate the supposed dunces from the enlightened. It is put on the same level as “the *fact* that the earth is round (a sphere) and not flat” with the dunces labeled “flat-earthers.”

However, I take umbrage with the use of the word “fact” to conflate these two instances as examples of what “is” or what is “true.” I claim the earth *is* spherical (more or less: a better approximation is an oblate spheroid) or certainly “curved” rather than “flat.” Whereas the statement that the “sun *is* at the center of the solar system” is not a fact but an arbitrary convenience. Try locating planets and constellations in the night sky using a heliocentric diagram. All sky maps are earth-centric and use earth-centric coordinates (declination and right-ascension). In fact, the designations of latitude and longitude (in Greek) seem to have originated with the Greek mathematician and astronomer Hipparchus (c.190 – c.120 BC) through his efforts to map the stars and planets on the celestial sphere. He then applied these methods to locate places on the “spherical” earth, which the Greeks knew all about (see [1]).

Heliocentrism

Probably the best demonstration of the significance of the heliocentric representation of the solar system versus the earth-centric is a video produced by the ebullient Nakul Dawra at his website GoldPlatedGoof ([2]). What it boils down to is really just two diagrams:

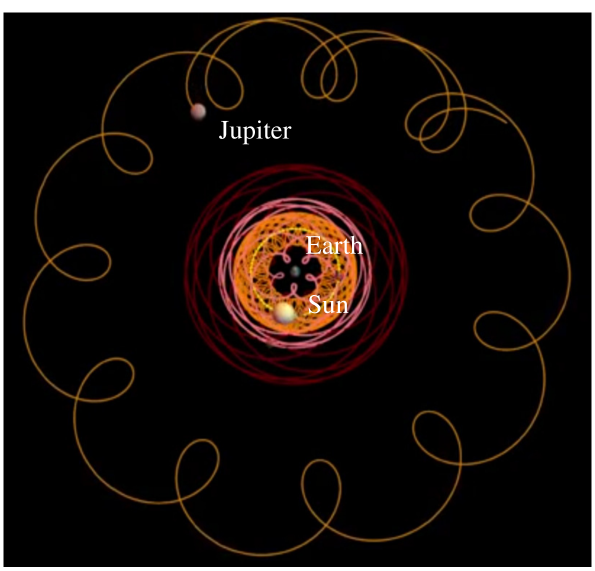

Figure 1 Earth-Centered

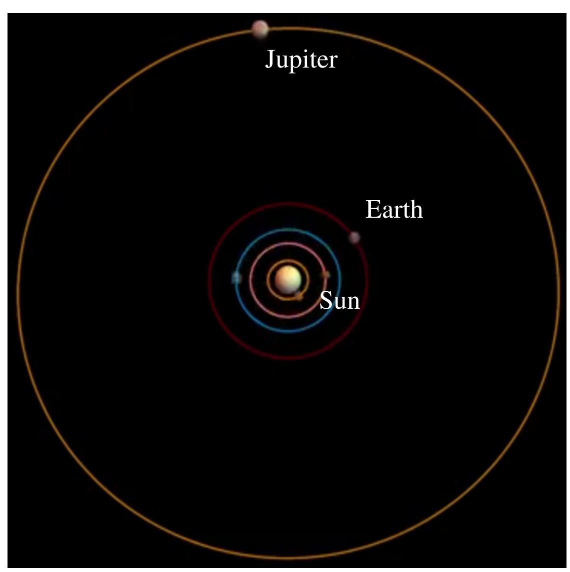

Figure 2 Helio-centric

Which diagram looks simpler? As we would say post-17th century Descartes, just by changing the origin of the polar coordinate system from the earth (Figure 1) to the sun (Figure 2) yields a more comprehensible diagram from which we can more easily compute the positions of the planets. The change of coordinate system does not change anything physically. The planets still do their thing. It isn't any more *true* that they revolve around the sun than that they revolve around the earth. It is just a mathematical change of perspective that is convenient. In fact, Tycho Brahe had a third representation of the solar system where all the planets except the earth revolved around the sun and then that whole system revolved around the earth. In his *Astronomia Nova* (1609) Kepler brilliantly showed Ptolemy's earth-centered solar system, Tycho Brahe's hybrid system, and Copernicus's heliocentric system (all involving circular orbits) were mathematically equivalent.

Thony Christie recently discussed how the heliocentric viewpoint was spread in the face of the supposed imprimatur of the Roman Catholic Church ([3]):

> First off, although *De revolutionibus* was placed on the Index in 1616, it was only placed there until corrected. In fact, somewhat against the norm, it was actually corrected surprisingly quickly and, with a few rather minor changes, became freely available again for Catholic scholars by 1621. … The only changes were that *any statements of the factual truth of the hypothesis were removed* … [my emphasis]

The heliocentric viewpoint could be freely discussed so long as it was not asserted to be a fact. But we have just seen this should not have been a problem. Where the difficulty came is that people still considered it a *hypothesis*, something that *might* be true or a fact. As I have said, this is the wrong way to view the matter, and a lot of historical grief could have been avoided if a more mathematical stance had been taken (difficult to do before the invention of coordinate systems by Fermat and Descartes).

The heliocentric situation is rather similar to the story of the "Coriolis force", which is really not a force, but the result of changing coordinate systems again. The alleged "force" comes from the law of inertia. Suppose we are standing at the center of a large, flat disk rotating counterclockwise at a constant angular rate $d\theta/dt$ (Figure 3). Suppose further that we have placed a large iron ball on a magnetized plate on the disk a distance r from the center so that it is "immobile." Then as we turn with the turning disk, we will see that the ball remains "fixed" to the disk at a distance r from us. However, from an elevated camera looking down on the disk, the iron ball will appear to be moving along the green path at a constant tangential rate of $v = r\, d\theta/dt$

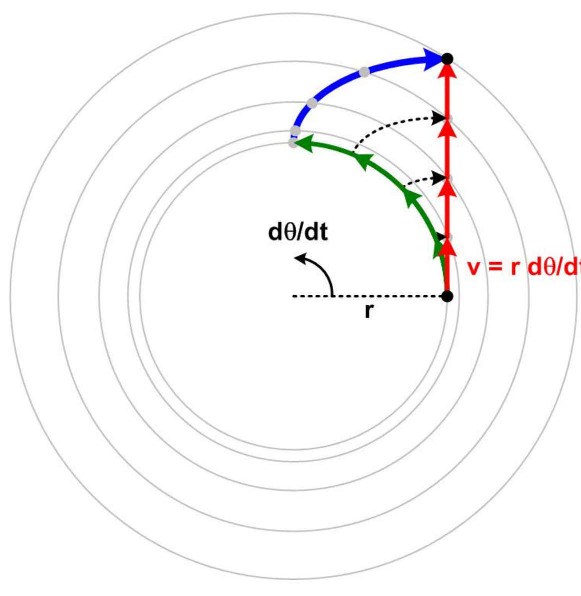

Figure 3 Coriolis "Force"

Now suppose the magnetic plate is suddenly demagnetized. Then the iron ball will no longer be constrained and, according to the Law of Inertia, will move off in a straight line at a constant speed of $v = r\, d\theta/dt$, as seen by the overhead camera (red path in Figure 3). From our vantage point at the center of the turning disk, we will see the ball moving away from us at an accelerating speed along the blue path to the right. And the path is even curving. So the combination of acceleration and curving will suggest to us that a force is acting to generate that motion, the so-called "Coriolis force." But there *is* no force—it is just a change in coordinate system, a change in perspective. Think of that the next time you are on a merry-go-round and try moving about. The "force" on your head is really fictitious. The real force is the friction of the merry-go-round on your shoes that forces them to go around while your head goes "straight ahead".

Curvature

We have discussed this concept in detail before ([4],[5], and also see *Wikipedia* [6]). The (Gaussian) curvature of a surface at a point is the product of its two principal curvatures k_1 and k_2 ($k_1 k_2$), which are associated with two perpendicular directions: one in the direction of maximal curvature and the other in the direction of minimal curvature. Each curvature is determined by a plane through the tangent to the surface in the given direction and the fixed normal to the surface at the given point. The "curvature" is then measured as the curvature of the curve formed by the intersection of the plane with the surface, specifically the curvature of an osculating circle at that point, namely the reciprocal of its radius (called the radius of curvature). For a perfect sphere every curve determined by the intersection of a plane through a normal to the sphere (and therefore through the center of the sphere) is a "great" circle with radius r, the radius of the sphere. So the two principal curvatures are in fact equal ($k_1 = k_2 = 1/r$) and the curvature of the sphere is $k_1 k_2 = 1/r^2$. In a flat plane, every intersection of a plane through a normal to the plane yields a straight line, which has zero curvature. (This can be thought of as a circle with infinite radius.) So a sphere (and even an oblate spheroid) has positive curvature, whereas a flat plane has zero curvature. (Of course, if we take the radius of a spherical earth to be approximately 4000 miles, then the Gaussian curvature would be $(1/4000)^2 = 0.0000000625 \approx 0$.)

The great *Theorema Egregium* of Gauss says that Gaussian curvature is invariant under local isometries. That is, if you bend a surface arbitrarily without tearing or stretching it (preserve distances locally), then it maintains the same Gaussian curvature. Therefore, there is no way to map the "spherical" earth onto a flat surface without causing distortions of some kind. And so it is a *fact* that the earth *is* not flat.

So there we have it. Heliocentrism is point of view, a change of coordinates, not a fact, whereas the earth being curved is a fact.

Muddying the Waters

I can't resist muddying the waters, which is what happens when mathematics sets out to "clarify" things. By the definition of Gaussian curvature above a cylinder has Gaussian curvature zero and so is "flat"! Because, pass a plane through the axis of the cylinder and its intersection with the cylinder is a straight line with curvature $k_1 = 0$. Pass a plane perpendicular to the axis and its intersection with the cylinder is a circle with cylindrical radius r and so curvature $k_2 = 1/r > 0$. But the Gaussian curvature is then $k_1 k_2 = 0 \cdot 1/r = 0$.

So now the flat-earther can say "I *see* that the earth is flat, and I *see* that the cylinder is curved. So are you telling me I should believe you rather than my lying eyes?"

The explanation is, of course, that mathematicians have defined a number of "curvatures." When we think about the curvature of a surface, we think about it curved *in space*, that is, as a 2-dimensional object embedded in a 3-dimensional space. The amazing and important thing about Gaussian curvature is that it is *intrinsic*, that is, it is independent of any embedding in a higher-dimensional space. It is what a creature living inside the "2-dimensional" curved surface would measure as its curvature. It is the 3-dimensional curvature of our 3-space that we would measure, since we cannot step into a 4-dimensional space to view it.

Now there are other definitions of curvature that depend on the embedding. They are called *extrinsic* curvatures. One of them is the mean curvature $(k_1 + k_2)/2$. So the cylinder has mean curvature $(0 + 1/r)/2 = 1/2r > 0$, and so is not flat relative to the mean curvature.

Of course, these ideas are getting rather abstract, and it is not surprising that the flat-earther would decide the mathematician's explanation is ridiculous. So I guess railing against the "fact" of heliocentrism and the belief of the flat-earthers is a fool's errand, but at least it is diverting.

References

[1] Stevenson, James, "Columbus and the Irony of Chance," *Meditations on Mathematics*, 29 March 2019, (http://josmfs.net/2019/03/29/columbus-and-the-irony-of-chance/)

[2] Dawra, Nakul, "Jupiter as the CENTER of the Solar System???" 18 December 2016 (https://www.youtube.com/watch?v=OULUNp8GkJQ). Dawra's 18 month silence is much lamented.

[3] Christie, Thony, "The Emergence Of Modern Astronomy – A Complex Mosaic: Part XXXII," *The Renaissance Mathematicus*, March 18, 2020 (https://thonyc.wordpress.com/2020/03/18/the-emergence-of-modern-astronomy-a-complex-mosaic-part-xxxii/)

[4] Stevenson, James, "Degree of Latitude," *Meditations on Mathematics*, 28 December 2018 (http://josmfs.net/2018/12/28/degree-of-latitude/)

[5] Stevenson, James, "Bugles, Trumpets, and Beltrami," *Meditations on Mathematics*, 24 February 2019 (http://josmfs.net/2019/02/24/bugles-trumpets-and-beltrami/)

[6] "Curvature" *Wikipedia* (https://en.wikipedia.org/wiki/Curvature)