Hard Geometric Problem

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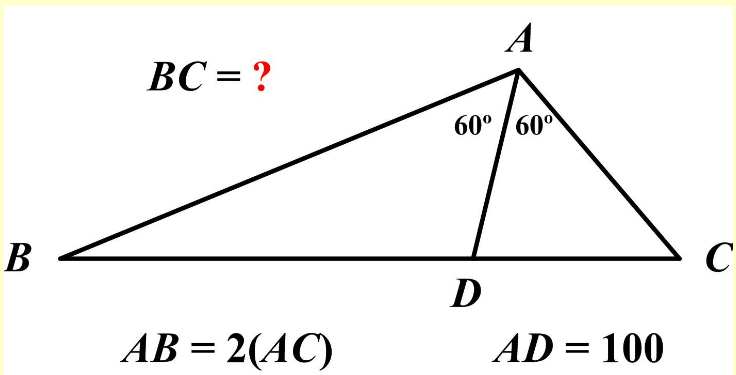
This is another problem from the indefatigable Presh Talwalkar.

(https://mindyourdecisions.com/blog/2019/09/09/hard-geometry-problem-from-switzerland/#more-32496, retrieved 10/3/2019)

Hard Geometry Problem From Switzerland

Posted September 9, 2019 By Presh Talwalkar.

This problem is adapted from a math contest in Switzerland—no calculators allowed.



In triangle *ABC* above, angle *A* is bisected into two 60° angles. If *AD* = 100, and *AB* = 2(*AC*), what is the length of *BC*?

Solution

I tried to solve the problem with the obvious law of cosines, but I couldn't unsnarl the results. It turned out that I also needed to use the (obscure to me) angle bisector theorem or the law of sines, which is how Talwalkar solved the problem (see below).

Instead, I noticed that the key points, B, C, D could all be represented by complex numbers via the modulus-argument form, as shown in Figure 1, where the length *r* (*AB*) was to be determined. The length *r* was like a scale factor that should be established via the length *AD* = 100.

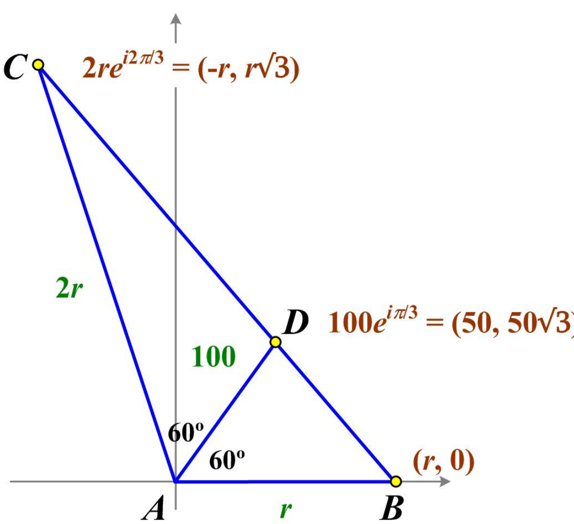


Figure 1 My Solution

First, I needed to convert the complex numbers to Cartesian coordinates:

$$100e^{i\frac{\pi}{3}} = 100(\cos 60 + i\sin 60) = 100(\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2})$$
$$= 50 + i50\sqrt{3}$$

and

$$2re^{i\frac{2\pi}{3}} = 2r(\cos 120 + i\sin 120) = 2r(-\tfrac{1}{2}) + i2r(\tfrac{\sqrt{3}}{2}) = -r + ir\sqrt{3}$$

Then the length *BC* is given by

$$BC = \sqrt{\left(0 - r\sqrt{3}\right)^2 + \left(r - (-r)\right)^2} = \sqrt{3r^2 + 4r^2} = \sqrt{7}r$$

Now the slope of *BD* is the same as the slope of *BC*, so using the Cartesian coordinates we have

$$\frac{0 - 50\sqrt{3}}{r - 50} = \frac{0 - r\sqrt{3}}{r - (-r)}$$

$$\frac{50}{r - 50} = \frac{1}{2}$$

$$r = 150$$

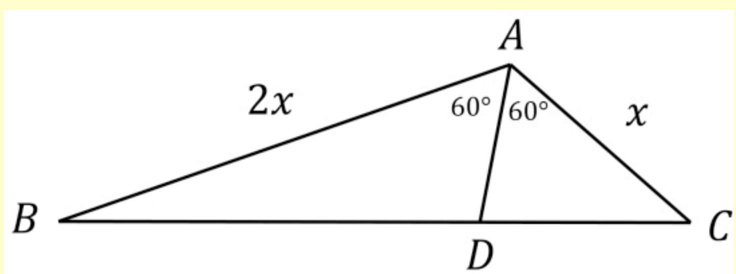
And so $BC = 150\sqrt{7}$

Talwalkar Solution

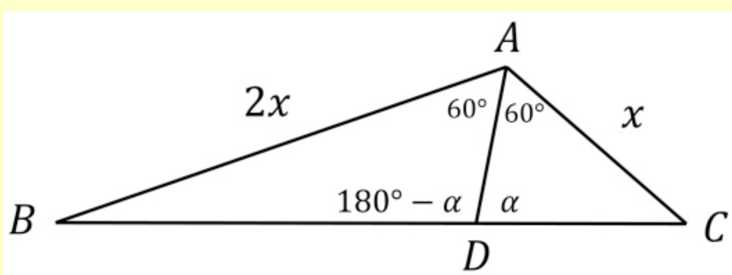
Talwalkar begins with a quote that I thought appropriate to include:

> "Anyone who knows, and knows that he knows,
> makes the steed of intelligence leap over the vault of heaven.
> Anyone who does not know but knows that he does not know,
> can bring his lame little donkey to the destination nonetheless.
> Anyone who does not know, and does not know that he does not know,
> is stuck forever in double ignorance."
>
> — Nasir al-Din al-Tusi

Let $AC = x$ so then $AB = 2x$.



Let angle $ADC = \alpha$ so then angle $ADB = 180° - \alpha$.



Now we will show $BD = 2(DC)$. This immediately follows from the **angle bisector theorem**, but let us prove it.

In triangle ADC, we will use the **law of sines**—which I read was first described by the Persian mathematician al-Tusi. (Perhaps we should call it Al-Tusi's law of sines).

We get:

$$DC/\sin 60° = x/\sin \alpha$$

Now we use the law of sines in triangle ADB to get:

$$BD/\sin 60° = 2x/\sin (180° - \alpha)$$

Now recall $\sin (180° - \alpha) = \sin \alpha$, so we have:

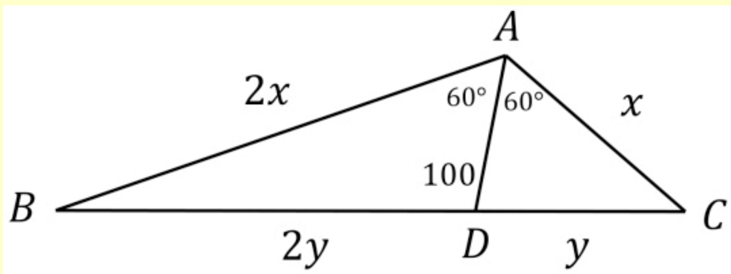
$$BD/\sin 60° = 2x/\sin \alpha$$

Hence we can divide the two equations to get:

$$BD/DC = 2$$

$$BD = 2(DC)$$

Now suppose $DC = y$ so $BD = 2y$.



Using Al-Kashi's **law of cosines** in triangle ADC we get:

$$y^2 = x^2 + 100^2 - 2(x)(100)\cos 60°$$

$$y^2 = x^2 + 100^2 - 100x$$

Similarly, using Al-Kashi's law of cosines in triangle ADB we get:

$$(2y)^2 = (2x)^2 + 100^2 - 2(2x)(100)\cos 60°$$

$$4y^2 = 4x^2 + 100^2 - 200x$$

So we have a system of two equations:

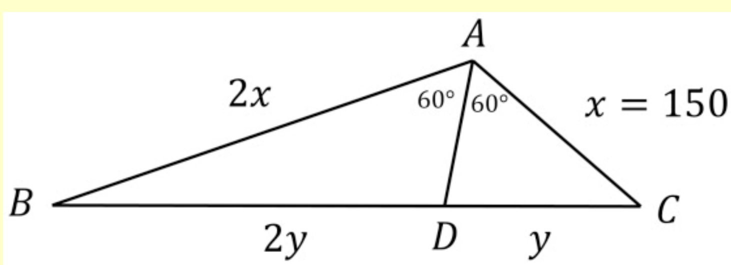
$$y^2 = x^2 + 100^2 - 100x$$

$$4y^2 = 4x^2 + 100^2 - 200x$$

If we multiply the first equation by 4 and then subtract the second we get:

$$0 = 3(100^2) - 200x$$

$$x = 150$$



Now we use Al-Kashi's law of cosines to solve for y in triangle ADC:

$$y^2 = 150^2 + 100^2 - 100(150)$$

$$y^2 = 17500$$

Since lengths are positive, we take the positive root:

$$y = 50\sqrt{7}$$

Finally we want to solve for $BC = 3y$, so we get $BC = 150\sqrt{7}$.

It's a pretty neat problem, and it's fun that it can be solved without a calculator!

Source

Adapted from FJSM 2019 semi-final (the problem was emailed to me)

http://www.smasv.ch/de/alle_aufgaben.php

Tusi mathematician

https://en.wikipedia.org/wiki/Nasir_al-Din_al-Tusi

http://www-history.mcs.st-andrews.ac.uk/Biographies/Al-Tusi_Nasir.html

Angle bisector theorem

https://en.wikipedia.org/wiki/Angle_bisector_theorem