Perpetual Meetings Problem

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The following problem from *Five Hundred Mathematical Challenges* ([1]) was a challenge indeed, even though it appeared to be a standard travel puzzle.

Problem 118. Andy leaves at noon and drives at constant speed back and forth from town A to town B. Bob also leaves at noon, driving at 40 km per hour back and forth from town B to town A on the same highway as Andy. Andy arrives at town B twenty minutes after first passing Bob, whereas Bob arrives at town A forty-five minutes after first passing Andy. At what time do Any and Bob pass each other for the n^{th} time?

My Solution

In Figure 1 d represents the distance in kilometers between towns A and B. d_1 is the distance from B when Andy and Bob first meet at time T_1. Therefore from the diagram we have

$$d_1 = r\,(1/3) = 40\,T_1$$

$$d - d_1 = r\,T_1 = 40\,(3/4) = 30 \text{ km}$$

Hence,

$$T_1 = r / 120 = 30 / r$$

$$r^2 = 3600$$

$$r = 60 \text{ kph}$$

$$T_1 = ½ \text{ hour}$$

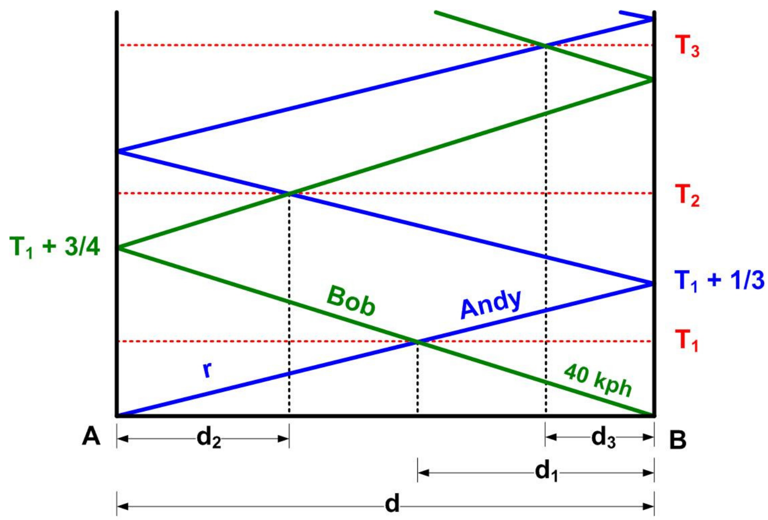

Figure 1 Problem Statement Space-Time Diagram

From the diagram it looks like Bob and Andy meet at regular intervals, that is, that the time intervals between their meetings are a constant. We need to prove that, and also find out what the constant time interval is.

$$40(T_2 - T_1) = (d - d_1) + d_2 = d + (d_2 - d_1)$$

$$r(T_2 - T_1) = d_1 + (d - d_2) = d - (d_2 - d_1)$$

Therefore, $$(40 + r)\,(T_2 - T_1) = 2d$$

Similarly, $$40(T_3 - T_2) = (d - d_2) + d_3 = d + (d_3 - d_2)$$

$$r(T_3 - T_2) = d_2 + (d - d_3) = d - (d_3 - d_2)$$

So $$(40 + r)\,(T_3 - T_2) = 2d$$

which implies $$(T_3 - T_2) = (T_2 - T_1)$$

It then follows, repeating the pattern, that all the intervals are equal.

Now from the first two equations above we get

$$(40 + r)\, T_1 = d$$

Therefore, from $$(40 + r)\,(T_2 – T_1) = 2d$$

we get $$2\, T_1 = (T_3 – T_2) = (T_2 – T_1) = \ldots$$

Since T_1 = ½ hour, all the intervals are 1 hour. Therefore, the n^{th} meeting time T_n is given by

$$T_n = T_1 + (n – 1)(2T_1) = ½ + (n – 1) \text{ hours}$$

500 Math Challenges Solution

First Solution. More generally, let v_a, v_b denote speeds of Andy and Bob respectively, and let t_a, t_b denote the time it takes Andy and Bob to first reach towns B and A, respectively, after they first pass each other. The progress of the two drivers can be illustrated by the "world-line" diagram of Figure 2.

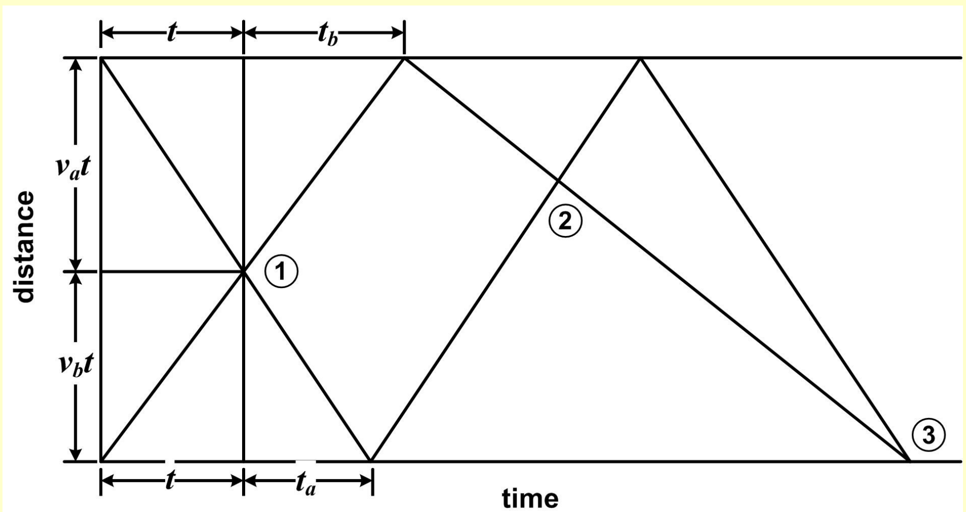

Figure 2 500 Math Challenges Solution

If t is the time elapsing between noon and their first passing, it follows from the diagram that

$$v_a = \frac{v_b t}{t_a}, \qquad v_b = \frac{v_a t}{t_b}$$

and hence

$$t = \sqrt{t_a t_b}\,, \qquad \frac{v_a}{v_b} = \sqrt{\frac{t_b}{t_a}}$$

For the data of the problem, t = 30 minutes and v_a = 60 kph.

To find the subsequent times of meeting, just keep on extending the world lines of Andy and Bob as indicated. One then obtains a periodic pattern after five passings which is centro-symmetric about the third passing, which occurs at town B. It follows that the n^{th} time of passing is 30 + 60(n – 1) minutes past noon.[1]

[1] JOS: I confess I don't see how this argument proves the intervals between each meeting are constant and equal to 60 minutes (twice the time of the first meeting).

Second Solution. The conditions of the problem yield the equations

$$(t + 20)v_a = (t + 45)40 = (v_a + 40)t = 60\text{d},$$

where d is the distance in kilometers between A and B, v_a is Andy's speed in kph, t is the time to first passing in minutes and the factor 60 relates the units of time. This gives $20v_a = 40t$ and $v_a t = 45{\cdot}40$, whence $v_a = 2t$ and $2t^2 = 45{\cdot}40$. Thus $t = 30$ minutes, $v_a = 60$ kph and $d = 50$ kilometers.[2]

After the first meeting, the cars are separating at a relative velocity of 100 kph for 20 minutes until Andy arrives at B, and then approaching at a relative velocity of 20 kph for 25 minutes until Bob arrives at A. During this time they have separated to a distance of 100/3 kilometers, then approached to a distance of 25 kilometers. When Bob leaves A, he is approaching Andy traveling from B, with a relative velocity of 100 kph. Thus they will meet 15 minutes later. The total time between first and second meetings is 60 minutes.

A similar argument establishes that the third meeting takes place at B 60 minutes later, and, now by symmetry, subsequent meetings occur at 60 minute intervals. Thus the two pass for the n^{th} time at $30 + 60(n - 1)$ minutes past noon.[3]

References

[1] Barbeau, Edward J., Murray S. Klamkin, William O. J. Moser, *Five Hundred Mathematical Challenges*, Spectrum Series, Mathematical Association of America, Washington D.C, 1995

[2] JOS: This is essentially my first argument.

[3] JOS: I did not bother to follow the argument, but at least here they proved the meeting intervals were constant and equal to 60 minutes.