A Tidy Theorem

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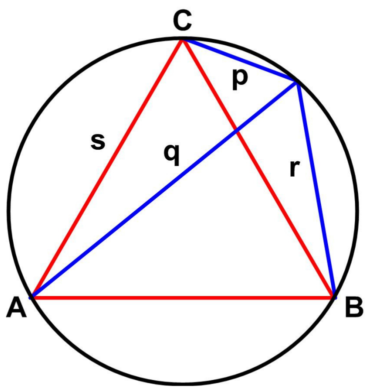

This is another fairly simple puzzle from *Futility Closet* ([1]).

If an equilateral triangle is inscribed in a circle, then the distance from any point on the circle to the triangle's farthest vertex is equal to the sum of its distances to the two nearer vertices ($q = p + r$).

(A corollary of Ptolemy's theorem.)

Proof

First, we see that the angles bounded by the blue lines are both 60° since they span the same arc of the circle as the 60° angles of the equilateral triangle (Figure 1).

Now it gets a bit messy. Since we are interested in lengths of a triangle given an included angle, it seems natural to consider the Law of Cosines and see what it yields.

$$s^2 = p^2 + q^2 - 2pq \cos 60° = p^2 + q^2 - pq$$

$$s^2 = r^2 + q^2 - 2rq \cos 60° = r^2 + q^2 - rq$$

$$p^2 - r^2 - q(p - r) = 0$$

$$p + r - q = 0 \qquad \text{if } p - r \neq 0$$

$$p + r = q$$

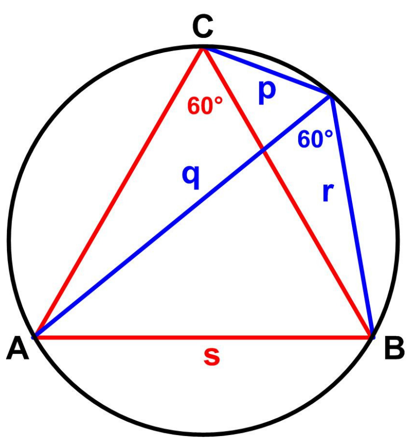

Figure 1 Problem Solution $p \neq r$

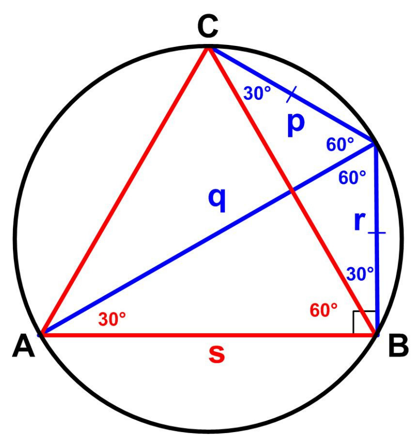

Figure 2 Problem Solution $p = r$

If $p = r$, then the concatenated blue triangles become an isosceles triangle with vertex angle 120° and so the base angles are 30° each. Therefore the angle at B is a right angle and we have 30-60 right triangle. Therefore, the hypotenuse q is twice the leg r, that is $q = 2r$. But $r = p$ means $q = p + r$, again.

References

[1] "A Tidy Theorem" *Futility Closet*, 14 March 2015 (http://www.futilitycloset.com/2015/03/14/a-tidy-theorem/, retrieved 2/5/16)