Quintic Nightmare

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Here is another challenging problem from the 2004 *Pi in the Sky* Canadian magazine for high school students ([1]).

Problem 4. Find the real solutions of the system

$$(x + y)^5 = z,$$

$$(y + z)^5 = x,$$

$$(z + x)^5 = y.$$

Solution

Let $f(x, y, z) = (x + y)^5 - z$. Notice that the three equations in the problem can all be obtained from $f(x, y, z) = 0$ by taking all permutations of the three variables x, y, z, where three pairs of the resulting six permuted equations are the same. Certainly 0, 0, 0 would be a solution to the equations, and the property of the permutations suggests that all solutions might be of the form $x = y = z$. (We now just quote the rest of the solution from *Pi in the Sky*.([2]))

Indeed, if for example we assume that $x < y$, then from the last two equations we get $(y + z)^5 < (z + x)^5$; hence $y < x$, which is a contradiction. Similarly, assuming any of the other possibilities results in contradictions.

Taking $x = y = z$ in the first equation, we get $(2x)^5 = x$; hence $x = 0$ or $x = \pm\frac{1}{2\sqrt{2}}$. Therefore the solutions of the system are

$$(0, 0, 0), \left(\tfrac{1}{2\sqrt{2}}, \tfrac{1}{2\sqrt{2}}, \tfrac{1}{2\sqrt{2}}\right), \left(-\tfrac{1}{2\sqrt{2}}, -\tfrac{1}{2\sqrt{2}}, -\tfrac{1}{2\sqrt{2}}\right)$$

References

[1] “Math Challenges,” *Pi in the Sky*, Issue 8, December 2004

[2] “Math Challenges,” *Pi in the Sky*, Issue 9, December 2005