Euler Magic

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$$\frac{e}{\sqrt{e}} \cdot \frac{\sqrt[3]{e}}{\sqrt[4]{e}} \cdot \frac{\sqrt[5]{e}}{\sqrt[6]{e}} \cdots = ?$$

This is a delightful and surprising problem from Presh Talwalkar.[1]

This puzzle was created by a MindYourDecisions fan in India. What is the value of the infinite product? The numerators are the odd n^{th} roots of [Euler's constant] e and the denominators are even n^{th} roots of e.

Solution

At first this just seems like a nightmare involving the ubiquitous Euler's constant e and a nest of n^{th} roots of an irrational number. So one just plods ahead, following their nose, in hopes something will fall out. It seems reasonable to convert the radicals to fractional exponents and perform the resulting manipulations:

$$(e/e^{1/2})(e^{1/3}/e^{1/4})(e^{1/5}/e^{1/6}) \ldots = e^1\, e^{-1/2}\, e^{1/3}\, e^{-1/4}\, e^{1/5}\, e^{-1/6} \ldots = e^{1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + \ldots}$$

The exponent is the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$$

and is known to converge, but to what?

Either you recall the power series for $\ln(1 + x)$ or it is easy to derive from our ever-trusty geometric series:

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt = \int_0^x \frac{1}{1-(-t)}\,dt = \int_0^x (1 - t + t^2 - t^3 + \ldots)dt = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots,$$

which converges for $-1 < x < 1$ because the geometric series does, and also converges for $x = 1$ by virtue of being an alternating series with the n^{th} term converging monitonically to zero. Therefore, setting $x = 1$ yields

$$\ln(1+1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \ln(2)$$

And so

$$e^{\ln(2)} = 2$$

Amazing—and wonderful! A problem which Talwalkar dubbed "too good"!

[1] https://mindyourdecisions.com/blog/2020/01/02/the-answer-is-too-good/