Amazing Triangle Problem

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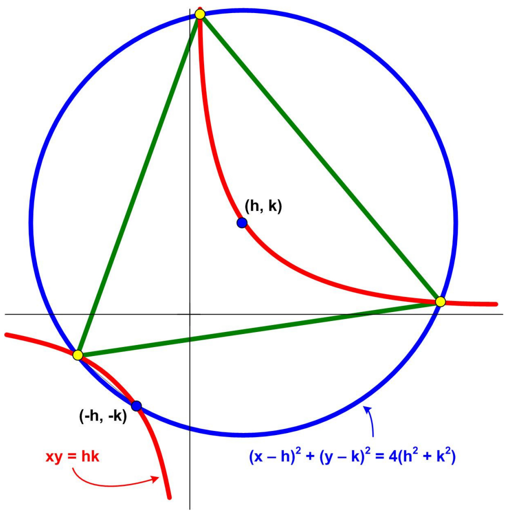

Here is another simply amazing problem from *Five Hundred Mathematical Challenges* ([1]):

Problem 154. Show that three solutions, (x_1, y_1), (x_2, y_2), (x_3, y_3), of the four solutions of the simultaneous equations

$$(x-h)^2 + (y-k)^2 = 4(h^2+k^2)$$

$$xy = hk$$

are vertices of an equilateral triangle. Give a geometrical interpretation.

Again, I don't see how anyone could have discovered this property involving a circle, a hyperbola, and an equilateral triangle. It seems plausible when $h = k$, but it is not at all obvious for $h \neq k$. For some reason, I had difficulty getting a start on a solution, until the obvious approach dawned on me. I don't know why it took me so long.

My Solution

The key is to parameterize the coordinates of the point (x, y) on the circle with trig functions (Figure 1). (I wasted too much time with Cartesian coordinates and also looking for geometric symmetries—the hyperbola kept being a ringer.)

$$x = 2r\cos\theta + h$$

$$y = 2r\sin\theta + k$$

Then (x, y) being a point of intersection of the circle with the hyperbola means

$$hk = xy = (2r\cos\theta + h)(2r\sin\theta + k)$$

or

$$\sin 2\theta = -\left(\frac{k}{r}\cos\theta + \frac{h}{r}\sin\theta\right)$$

$$= -(\sin\alpha\cos\theta + \cos\alpha\sin\theta)$$

$$= -\sin(\alpha+\theta)$$

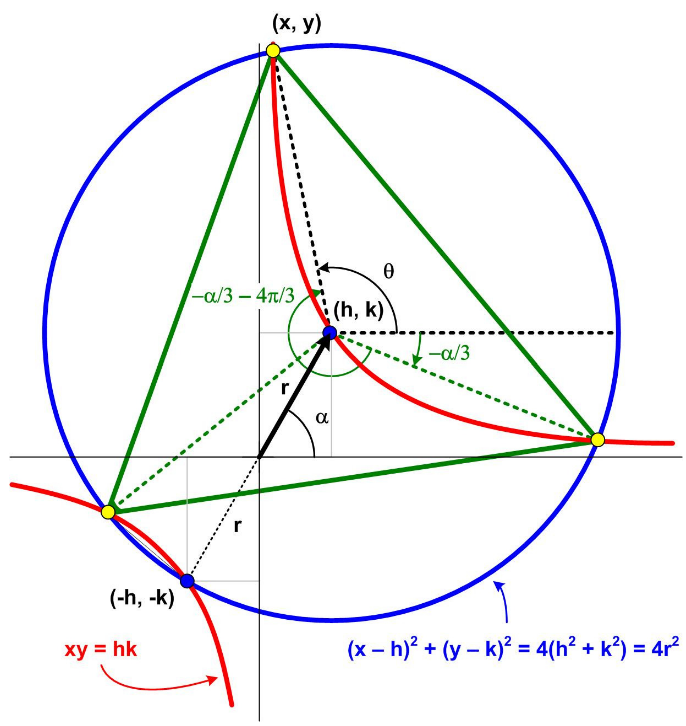

Figure 1 Problem Parameterization and Solution

Therefore,

$$-2\theta = \alpha + \theta + 2n\pi \qquad \text{for } n = 0, 1, 2, \ldots,$$

or

$$\theta = -\frac{\alpha}{3} - \frac{2}{3}n\pi \qquad \text{for } n = 0, 1, 2$$

where 2π/3 is 120°, thus producing vertices at a constant distance $2r$ from the center of the circle and separated by 120°—an equilateral triangle.

500 Math Challenges Solution

Given the existence of the circle in the problem, 500 Math Challenges also effectively parameterized by trig functions, only in the form of complex variables. Here is their solution.

Let $h = r\cos\theta$,[1] $k = r\sin\theta$, $x - h = 2ru$, $y - k = 2rv$. Then the two equations become

$$u^2 + v^2 = 1,$$

$$(2u + \cos\theta)(2v + \sin\theta) = \sin\theta\cos\theta.$$

Let $z = u + iv$, so that

$$\bar{z} = u - iv,\ 2u = z + \bar{z},\ 2v = \frac{z - \bar{z}}{i},$$

$$z\bar{z} = 1 \tag{1}$$

and (with cis $\theta = \cos\theta + i\sin\theta$)

$$z(z + cis\theta) = \bar{z}(\bar{z} + cis(-\theta))\,. \tag{2}$$

Multiplying (2) by z^2 and using (1) yields

$$z^3(z + cis\theta) = 1 + z\,cis(-\theta),$$

which can be manipulated to give

$$(z^3 - cis(-\theta))(z + cis\theta) = 0\,.$$

The root z = –cis θ corresponds to

$$x = h + 2ru = h - 2r\cos\theta = -h.$$

and $$y = k + 2rv = k - 2r\sin\theta = -k,$$

the coordinates of the point (-h, -k). The other factor, $z^3 - \text{cis}(-\theta)$, has three roots:

$$z = cis\frac{-\theta}{3}, \qquad \text{i.e.,}\ u = \cos\frac{\theta}{3}, \qquad v = -\sin\frac{\theta}{3};$$

$$z = cis\frac{-\theta + 2\pi}{3}, \qquad \text{i.e.,}\ u = \cos\frac{\theta - 2\pi}{3}, \qquad v = -\sin\frac{\theta - 2\pi}{3};$$

$$z = cis\frac{-\theta - 2\pi}{3}, \qquad \text{i.e.,}\ u = \cos\frac{\theta + 2\pi}{3}, \qquad v = -\sin\frac{\theta + 2\pi}{3};$$

These roots give the points of intersection

[1] JOS: Here θ is the same as my α.

$$\left(h+2r\cos\frac{\theta}{3},k-2r\sin\frac{\theta}{3}\right),$$

$$\left(h+2r\cos\frac{\theta-2\pi}{3},k-2r\sin\frac{\theta-2\pi}{3}\right),$$

$$\left(h+2r\cos\frac{\theta+2\pi}{3},k-2r\sin\frac{\theta+2\pi}{3}\right),$$

which are the vertices of an equilateral triangle (Figure 2).

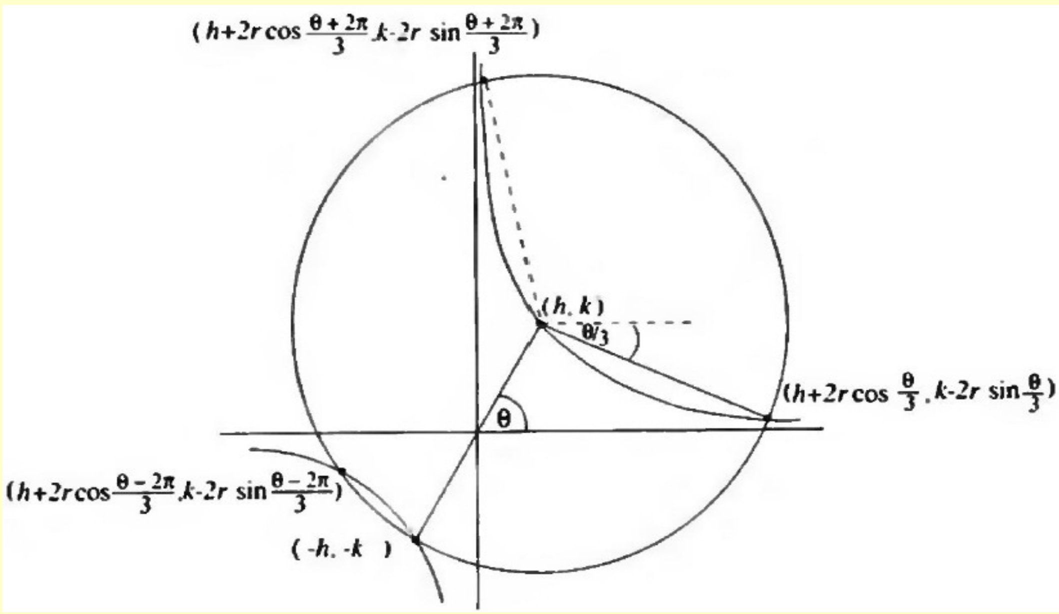

Figure 2 Solutions to Simultaneous Equations

For a geometrical interpretation, take any point (h, k) on an equilateral hyperbola $xy = a^2$. Construct a circle with center (h, k) and passing through ($-h$, $-k$). This circle will intersect the hyperbola in three other points which are the vertices of an equilateral triangle.

Comment

The existence of this equilateral triangle is slightly reminiscent of a posting from Fermat's Library ([2]):

> This question was asked by Toeplitz in 1911 and remains unsolved to this day: Every simple closed [non-intersecting] curve that you can draw by hand [continuous] will pass through the corners of some square (Figure 3).

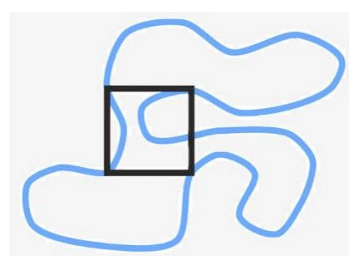

Figure 3

Certainly a circle and intersecting hyperbola are more specific than a general, non-self-intersecting continuous curve, but the equilateral triangle result is still surprising in its own right.

References

[1] Barbeau, Edward J., Murray S. Klamkin, William O. J. Moser, *Five Hundred Mathematical Challenges*, Spectrum Series, Mathematical Association of America, Washington D.C, 1995

[2] Fermat's Library, 10 December 2019 (https://twitter.com/fermatslibrary/status/1204400428554838021, retrieved 12/12/2019)