Magic Hexagons

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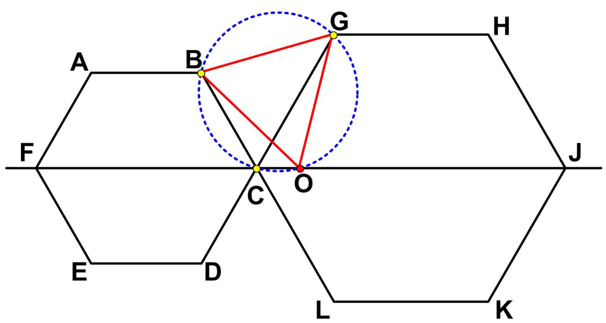


This is truly an amazing result from *Five Hundred Mathematical Challenges* ([1]).

Problem 119. Two unequal regular hexagons ABCDEF and CGHJKL touch each other at C and are so situated that F, C, and J are collinear. Show that

(i) the circumcircle of BCG bisects FJ (at O say);

(ii) ΔBOG is equilateral.

I wonder how anyone ever discovered this.

My Solution

Again it appears my approach was more complicated, though it was straight-forward.

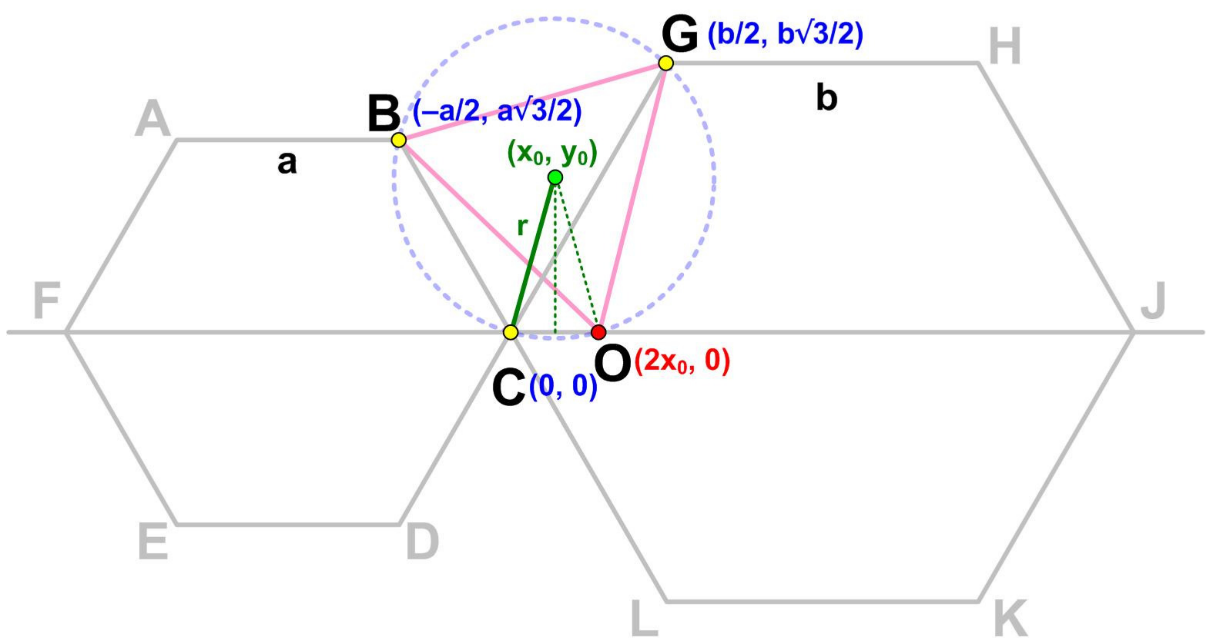


Figure 1 My Solution

If we let a be the length of the side of the small hexagon and b the length of the side of the large hexagon, and if we let C be the origin of a coordinate system (Figure 1), then the other two points on the circumcircle B and G have coordinates (-a/2, a√3/2) and (b/2, b√3/2). Since the points C and O lie on the horizontal line FJ, the perpendicular bisector of the line joining them will pass vertically through the center of the circle given by coordinates (x_0, y_0). Therefore O lies a distance $2x_0$ from C.

Three points are sufficient to determine a circle and therefore its equation. We use the center-radius form, which gives the locus as all points (x, y) a distance r from the center (x_0, y_0):

$$(x - x_0)^2 + (y - y_0)^2 = r^2$$

Substituting the coordinates for C, B, and G gives us three equations:

C (0, 0): $x_0^2 + y_0^2 = r^2$

B (–a/2, a√3/2): $(-a/2 - x_0)^2 + (a\sqrt{3}/2 - y_0)^2 = r^2$

G (b/2, b√3/2): $(b/2 - x_0)^2 + (b\sqrt{3}/2 - y_0)^2 = r^2$

This yields the pair of equations

$$-x_0 + \sqrt{3}y_0 = a$$

$$x_0 + \sqrt{3}y_0 = b$$

Therefore, $2x_0 = b - a$

Now $FJ = FC + CJ = 2a + 2b = 2(a + b)$

and $FO = FC + 2x_0 = 2a + (b - a) = a + b = FJ / 2$

Hence, O is the midpoint of the line FJ, proving (i).

We now compute the lengths of the edges of the triangle BG, BO, and OG.

$$BG^2 = (b/2 + a/2)^2 + (b\sqrt{3}/2 - a\sqrt{3}/2)^2 = a^2 - ab + b^2$$

$$BO^2 = ((b - a) + a/2)^2 + (-a\sqrt{3}/2)^2 = a^2 - ab + b^2$$

$$OG^2 = (b/2 - (b - a))^2 + (b\sqrt{3}/2)^2 = a^2 - ab + b^2$$

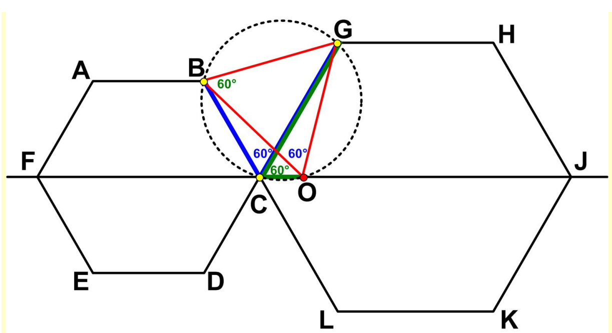
So all three sides are equal, making the triangle BGO equilateral, proving (ii).

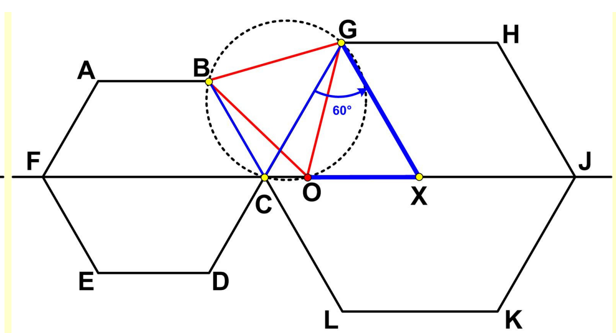
Notice that if the hexagons are the same size (a = b), then clearly O bisects FJ and the lengths BG, BO, and OG all reduce to a, as they should.

500 Math Challenges Solution

Naturally, they had a simpler solution. It is purely plane geometry rather than analytic geometry like my approach, and quite slick. (I have added my own diagrams for clarity.)

Since ∠BCG = ∠GCO = 60° and B, C, O, G are concyclic, it follows that ∠BOG = ∠GBO = 60° and hence triangle BGO is equilateral.[1]





Let X be the center of the larger hexagon. A counterclockwise rotation of 60° about G maps B and C onto O and X respectively. Hence, BC = OX, and

$$FO = FC + CO = 2BC + CO = 2OX + CO = OX + CX = OJ.$$

[1] JOS: ∠BOG = ∠GBO = 60° since ∠BOG = ∠BCG and ∠GBO = ∠GCO because they subtend the same arcs of the circle.

References

[1] Barbeau, Edward J., Murray S. Klamkin, William O. J. Moser, *Five Hundred Mathematical Challenges*, Spectrum Series, Mathematical Association of America, Washington D.C, 1995