Number of the Beast

24 November 2019

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If you will pardon the pun, this is a diabolical problem from the collection *Five Hundred Mathematical Challenges* ([1]).

Problem 5. Calculate the sum

$$6 + 66 + 666 + \ldots + \underbrace{666\ldots6}_{n} \qquad (n \geq 1)$$

It has a non-calculus solution, but that involves a bunch of manipulations that were not that evident to me, or at least I doubt if I could have come up with them. I was able to reframe the problem using one of my favorite approaches, power series (or polynomials). The calculations are a bit hairy in any case, but I was impressed that my method worked at all.

My Solution

The first thing I did was factor out the 6:

$$6 + 66 + 666 + \ldots + 666\ldots6 = 6(1 + 11 + 111 + \ldots + 111\ldots1) = 6\, S_n$$

Then I noticed S_n could be written

n	Original	As Powers of 10	Cumulative Sum S_n
1	1	1	1
2	11	$10 + 1$	12
3	111	$10^2 + 10 + 1$	123
4	1111	$10^3 + 10^2 + 10 + 1$	1234

$$S_4 = 10^3 + 2{\cdot}10^2 + 3{\cdot}10 + 4{\cdot}1$$

so

$$S_n = 1{\cdot}10^{n-1} + 2{\cdot}10^{n-2} + 3{\cdot}10^{n-3} + \ldots + (n-2)10^2 + (n-1)10 + n{\cdot}1$$

or factoring out 10^{n-1} and setting $x = 1/10$,

$$S_n = (1 + 2x + 3x^2 + 4x^3 + \ldots + nx^{n-1}) \,/\, x^{n-1}.$$

Thus the beginning of a power series starts appearing, which looks like the derivative of the geometric series. Let the geometric series partial sums be given by

$$f(x) = 1 + x + x^2 + x^3 + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

Then

$$S_n = f'(x) \,/\, x^{n-1}$$

Now

$$f'(x) = \frac{nx^{n+1} - (n+1)x^n + 1}{(1-x)^2},$$

so

$$S_n = \frac{f'(x)}{x^{n-1}} = \frac{nx^2 - (n+1)x + \dfrac{1}{x^{n-1}}}{(1-x)^2}.$$

Substituting $x = 1/10$ yields

$$S_n = \left(\frac{n}{10^2} - (n+1)\frac{1}{10} + 10^{n-1}\right)\frac{10^2}{9^2}$$

Therefore,

$$S_n = \frac{1}{9^2}\left(10^{n+1} - (9n+10)\right)$$

As a check, we compute S_n for $n = 1, 2, 3, 4$.

$S_1 = (100 - 19) / 9^2 = 81 / 9^2 = 9 / 9 = 1$

$S_2 = (1000 - 28) / 9^2 = 972 / 9^2 = 108 / 9 = 12$

$S_3 = (10000 - 37) / 9^2 = 9963 / 9^2 = 1107 / 9 = 123$

$S_4 = (100000 - 46) / 9^2 = 99954 / 9^2 = 11106 / 9 = 1234$

Thus the final formula is $6\,S_n = (2/27)\,(10^{n+1} - (9n + 10))$

I can't help it—I love this technique. It is no wonder that Newton based most of his early calculations on power series, since they in effect reduced operations with most of the common functions to manipulating polynomials.

500 Math Challenges Solution

Call the required sum S_n, so that for $n \geq 2$

$$S_n = 6 + 66 + 666 + \ldots + \underbrace{666\ldots6}_{n} = (0 + 6) + (60 + 6) + (660 + 6) + \ldots + (\underbrace{666\ldots6}_{n\text{-}1}0 + 6),$$

$$= 10\,(6 + 66 + 666 + \ldots + \underbrace{666\ldots6}_{n\text{-}1}) + 6n = 10\,S_{n\text{-}1} + 6n$$

$$= 10\,(S_n - \underbrace{666\ldots6}_{n}) + 6n$$

Solving for S_n we obtain

$$9S_n = \underbrace{666\ldots6}_{n}0 - 6n = (2/3)(999\ldots90 - 9n)$$

$$S_n = (2/3)(\underbrace{111\ldots1}_{n}0 - n)$$

Since

$$\underbrace{111\ldots1}_{n}0 = 10 + 10^2 + 10^3 + \ldots + 10^n = 10\,(10^n - 1) / 9,$$

we may write

$$S_n = (2/3)\,(10\,(10^n - 1) / 9) = (2/27)\,(10^{n+1} - 10 - 9n)$$

References

[1] Barbeau, Edward J., Murray S. Klamkin, William O. J. Moser, *Five Hundred Mathematical Challenges*, Spectrum Series, Mathematical Association of America, Washington D.C, 1995