Movie Projector Problem

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Here is another Brain Bogglers problem from 1987 ([1]).

Exactly four minutes after starting to run—when the take-up reel was rotating one and a half times as fast as the projecting reel—the film broke. (The hub diameter of the smaller take-up reel is 8 cm and the hub diameter of the projecting reel is 12 cm.) How many minutes of film remain to be shown?

This feels like another problem where there is insufficient information to solve it, and that makes it fun and challenging. In fact, I was stumped for a while until I noticed something that was the key to completing the solution.

Solution

Figure 1 shows the situation for the problem. One important constraint is that the film is spooling from the projecting reel to the take-up reel at a constant rate v_0. That means in a small interval of time Δt, the same amount of film Δs is wrapped around the take-up reel as leaves the projecting reel. The segment of arc Δs of a circle is given by the product of the radius and the small increment of angle subtended by the arc. Since the two reels of film are different sizes, the radii and angles will be different, and will also change over time. So we have

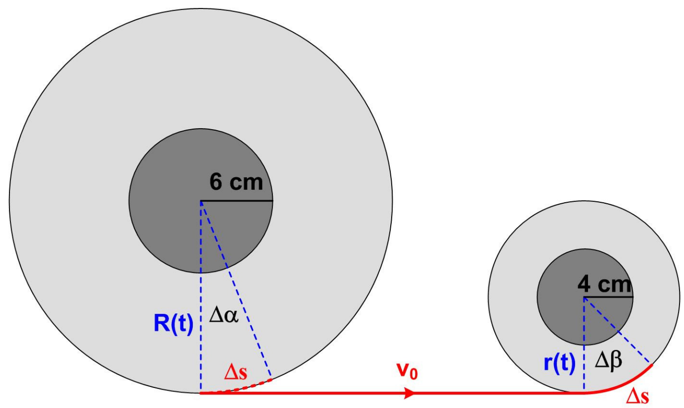

Figure 1 Reel to Reel

$$R(t)\,\Delta\alpha(t) = \Delta s = r(t)\,\Delta\beta(t)$$

Hence

$$R(t)\frac{\Delta\alpha}{\Delta t} = \frac{\Delta s}{\Delta t} = v_0 = r(t)\frac{\Delta\beta}{\Delta t}$$

Passing to the limit as $\Delta t \to 0$, we get the instantaneous speeds at t = 4 minutes.

$$R(4)\frac{d\alpha}{dt} = \frac{ds}{dt} = v_0 = r(4)\frac{d\beta}{dt}$$

We are told that at that instant, the ratio of the rotation rates of the two reels $(d\beta/dt) / (d\alpha/dt)$ is 3/2. Therefore, $R(4)/r(4) = 3/2$ also.

Now we need to come up with an expression involving time. Since the film spools at a constant rate, we need to consider the amount of film left on the projecting reel. And since the thickness and the width of the film is the same on both reels, the length of film (and therefore time) on both reels is proportional to the areas of the film on the two reels, say A_1 and A_2 for the projecting and take-up reels respectively. (Note that the areas are the annuli obtained from subtracting the areas of the hubs from the areas of the filled reels.) If T is the time remaining on the projecting reel, then

$$\frac{T}{4} = \frac{A_1}{A_2} = \frac{\pi(R^2 - 6^2)}{\pi(r^2 - 4^2)}$$

Now R = (3/2)r. Replacing R with r would still leave r to consider. But then I realized 6 = (3/2)4 as well and voila, terms in r disappear:

$$\frac{T}{4} = \frac{R^2 - 6^2}{r^2 - 4^2} = \frac{\frac{9}{4}(r^2 - 4^2)}{(r^2 - 4^2)} = \frac{9}{4}$$

Thus T = 9 minutes, the time remaining. Amazing!

References

[1] Stueben, Michael, "Brain Bogglers," *Discover*, March 1987