Geometric Puzzle Mayhem

9 November 2019

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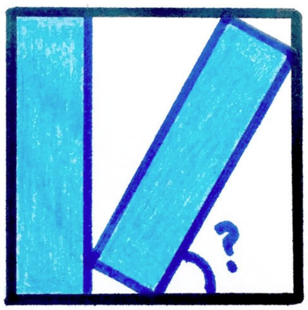
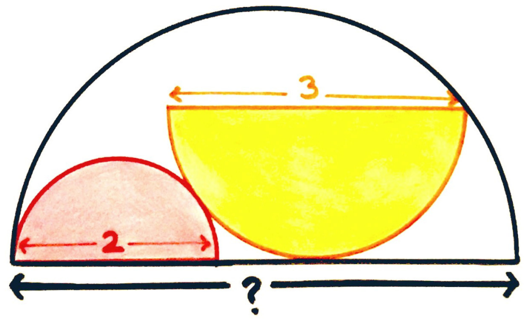
#3 Tilted Twin

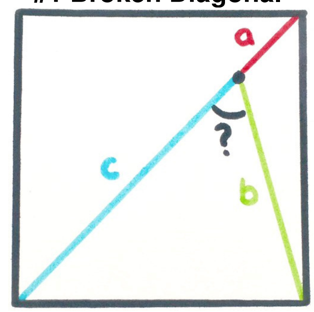


The blue rectangles are congruent. What's the angle?

#2 Bowls in Bowls



#1 Broken Diagonal

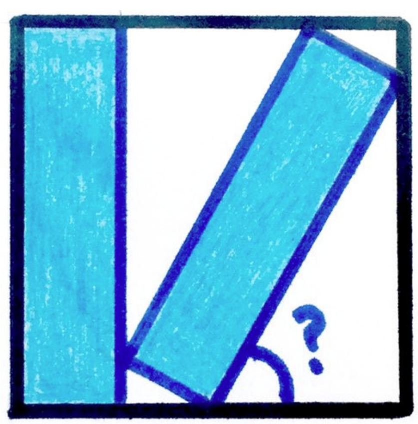


Point on diagonal. a + b = c. What's the angle?

I was really trying to avoid getting pulled into more addictive geometric challenges from Catriona Shearer (since they can consume your every waking moment), but a recent post by Ben Orlin, "The Tilted Twin (and other delights),"[1] undermined my intent. As Orlin put it, "This is a countdown of her three favorite puzzles from October 2019" and they are vintage Shearer. You should check out Olin's website since there are "Mild hints in the text; full spoilers in the comments." He also has some interesting links to other people's efforts. (Olin did leave out a crucial part of #1, however, which caused me to think the problem under-determined. Checking Catriona Shearer's Twitter[2] I found the correct statement, which I have used here.)

I have to admit, I personally found the difficulty of these puzzles a bit more challenging than before (unless I am getting rusty) and the difficulty in the order Olin listed. Again, the solutions (I found) are simple but mostly tricky to discover. I solved the problems before looking at Olin's or others' solutions.

Puzzle #3: The Tilted Twin



The blue rectangles are congruent. What's the angle?

[1] https://mathwithbaddrawings.com/2019/11/04/the-tilted-twin-and-other-delights/
[2] https://twitter.com/Cshearer41/status/1182594496938336257

My Solution

Figure 1 shows the solution where I have chosen (wlog[3]) a unit square and labeled the width of the blue rectangle x. The key for me was realizing that since the tilted rectangle had to touch both the top and bottom of the square with two of its vertices, that meant a diagonal had to as well, and perforce as a diagonal it had to be parallel to a diagonal of the untilted rectangle. Given the resulting symmetries of the problem, the spaces between the rectangles are correspondingly congruent and I have labeled the width of the identical empty rectangles y.

From the figure we see that

$$a^2 + b^2 = 1$$

But also b = x + y, which implies 2b = 1. Therefore b = 1/2 and a = √3/2, which means

$$\alpha = 60°.$$

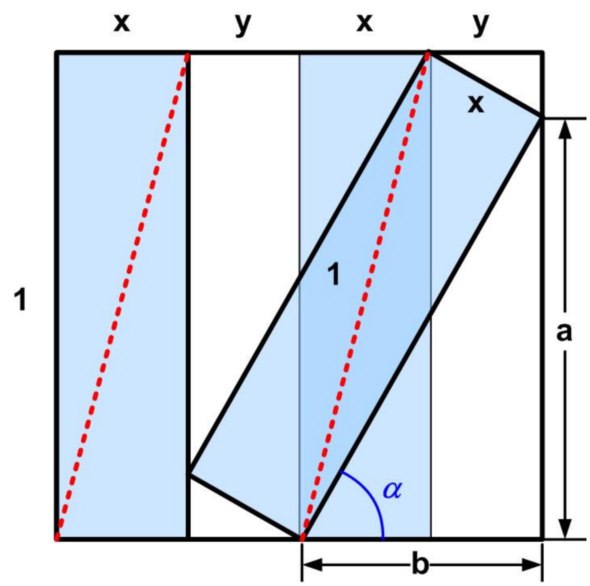
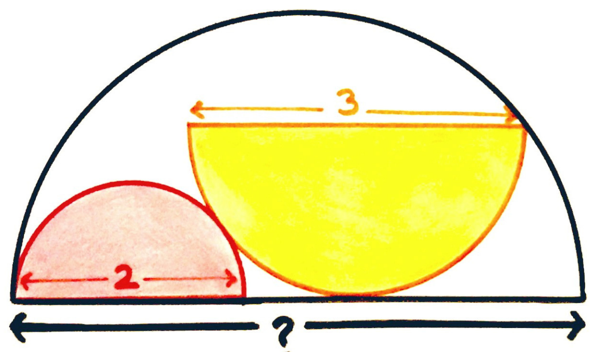


Figure 1 Tilted Twin Solution

Puzzle #2: Bowls in Bowls



My Solution

I solved the problem in two steps, the second taking the longest to find. Figure 2 shows the first step. The line between the centers of the two semicircles passes through the tangent point (LTR[4]) and so its length is the sum of the two radii: 1 + 3/2 = 5/2. Hence, we have a 3-4-5 right triangle, which implies the second leg is of length 4/2 = 2.

Now Figure 3 shows the second step where

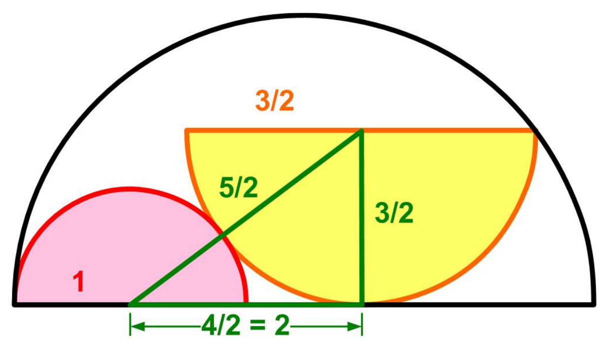


Figure 2 Bowls in Bowls Solution Step 1

[3] "without loss of generality"

[4] "Left to reader" This has been proved in other problems, such as in the "Kissing Angles" and in the "Circle Tangent Chord Problem."

we have joined the extremities of the colored semicircles that touch the large semicircle with a chord of the large semicircle. We then join the yellow semicircle vertex with a chord to the end of the diameter of the large semicircle, thus obtaining an inscribe right triangle. Using the Geometric Mean theorem we have

$$\frac{9/2}{3/2} = \frac{3/2}{x}$$

$$x = 1/2.$$

Therefore, the length of the diameter is 10/2 = 5.

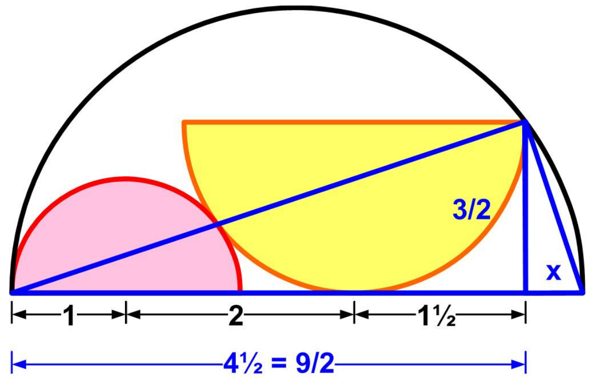
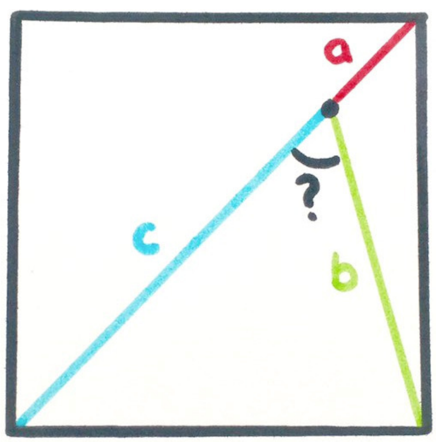


Figure 3 Bowls in Bowls Solution Step 2

What took me so long to find this solution was that I kept playing with the 3-4-5 triangle in Step 1. Using Visio, all sorts of fascinating results occurred with various rotations, reflections, translations, and dilations of the triangle, but I couldn't figure out how to *prove* them. I realized I hadn't used the fact that the yellow semicircle touched the large semicircle. That concentrated my thinking to arrive at Step 2 of the solution.

Puzzle #1: The Broken Diagonal



Point on diagonal. a + b = c. What's the angle?

My Solution

I found this problem, as originally stated in Olin's website, to be a bear and I wasted a day on analyzing it. I arrived at the conclusion that it was impossible. So I sought out Catriona Shearer's original statement, and much to my amazement and gratification there was an additional explicit constraint—*the point lay on the diagonal*!

I initially thought the intersection point was floating within the square and spent a great deal of time working out that it could be any point along an arc that looked very much like that of a circle with radius twice the edge of the square. But the unknown angle was not constant along that arc. Yes, the picture seems to show the point on the diagonal, but I try not to trust a picture and only use what is specified.

Now with the correct problem statement the solution is straight-forward. Again I considered a unit square (Figure 4). The unknown angle is made up of a $45° = \pi/4$ angle and a sliver designated θ. Then we have

$$\cos\theta = \frac{c/\sqrt{2}}{b} = \frac{1}{\sqrt{2}}\left(\frac{a+b}{b}\right)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{a}{b}+1\right) = \cos\left(\frac{\pi}{2}-\theta\right) + \frac{1}{\sqrt{2}}$$

$$= \sin\theta + \frac{1}{\sqrt{2}}$$

Therefore

$$\cos\theta - \sin\theta = 1/\sqrt{2}$$

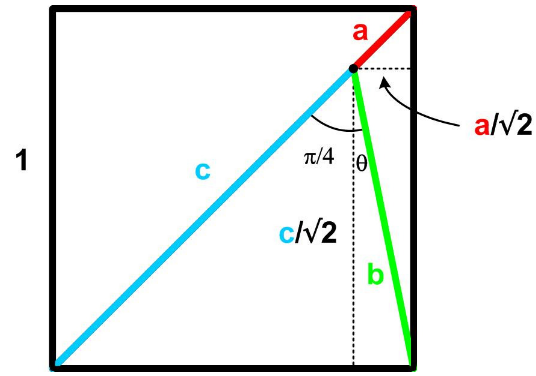


Figure 4 Broken Diagonal Solution

Now

$$\cos\left(\theta+\frac{\pi}{4}\right) = \cos\theta\cos\frac{\pi}{4} - \sin\theta\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}(\cos\theta - \sin\theta) = \frac{1}{2}$$

Therefore the unknown angle $\theta + \pi/4 = \pi/3 = 60°$.

Commentary

I have now looked at the links on Ben Olin's website.

In the first puzzle, Alaaddin Çizer noticed the parallelism of the second diagonal, but he proceeded to finish the problem differently.

In the second puzzle, Mary Pardoe seemed to fall into the trap I experienced with the 3-4-5 triangles (Figure 5). I also *saw* the vertex of the reflected 3-4-5 triangle land on the end of the diameter of the semicircle, but I could not *prove* it. I needed to use the fact that the inverted semicircle touched the large semicircle to complete the problem.

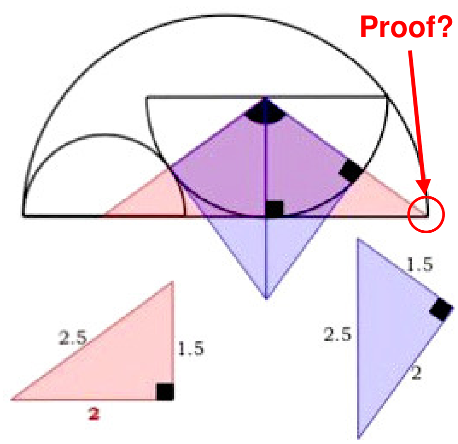


Figure 5 Mary Pardoe Solution

In the third problem David Butler showed a number of approaches that were basically equivalent to mine but utilized different parts of the figure.

I did not bother to check Catriona Shearer's Twitter further or the comments there.

As usual, great fun and stimulation. I still don't know how she comes up with her problems.