Unexpected Sum

1 August 2019

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This is another stimulating math problem from Colin Hughes's *Maths Challenge*[1] website (mathschallenge.net).

Problem

Find the exact value of the following infinite series:

1/2! + 2/3! + 3/4! + 4/5! + ...

My Solution

As usual, I see a series like

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+\ldots \tag{1}$$

and think of power series, in this case the power series for e^x, namely

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots$$

I then try to see if I can manipulate this power series to get the desired series (1):

$$e^x - 1 = \frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots = x\left(\frac{1}{1!}+\frac{x}{2!}+\frac{x^2}{3!}+\frac{x^3}{4!}+\ldots\right)$$

so

$$f(x) = \frac{e^x - 1}{x} = \frac{1}{1!}+\frac{x}{2!}+\frac{x^2}{3!}+\frac{x^3}{4!}+\ldots \tag{2}$$

means we can differentiate the power series for $f(x)$ term by term to get

$$f'(x) = \frac{1}{2!}+\frac{2x}{3!}+\frac{3x^2}{4!}+\frac{4x^3}{5!}+\ldots$$

and so $f'(1)$ yields the series (1). But the equation (2) also means

$$f'(x) = \frac{e^x(x-1)+1}{x^2}$$

and so $f'(1) = 1$. Therefore,

$$\boxed{\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+\ldots = 1}$$

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[1] "Unexpected Sum" Problem ID: 276 (21 April 2006) Difficulty: 3 Star at mathschallenge.net. "A three-star problem: a good knowledge of school mathematics and/or some aspects of proof will be required." (https://mathschallenge.net/problems/pdfs/mathschallenge_3_star.pdf)

Perhaps this seems too devious and too much trickery, but I consider this to be a general approach to problems like this, as I have shown in the past ("Infinite Product Problem"). It does mean you have to remember some basic power series, but that is standard operating procedure.

Maths Challenge Solution

Now I find the Maths Challenge solution tricky, especially since I find seeing that a series is a telescoping series to be more difficult than seeing how to manipulate basic power series, such as the geometric series or exponential series. Anyway, here is the Maths Challenge solution.

First we note that the general term in this series can be written differently:

$$\frac{k}{(k+1)!} = \frac{k+1-1}{(k+1)!} = \frac{1}{k!} - \frac{1}{(k+1)!}$$

Hence the original series becomes a telescoping series:

$$\boxed{\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \ldots = \left(\frac{1}{1!} - \frac{1}{2!}\right) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \left(\frac{1}{3!} - \frac{1}{4!}\right) + \ldots = 1}$$