Fibonacci, Chickens, and Proportions

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There is the famous chicken and the egg problem: If a chicken and a half can lay an egg and a half in a day and a half, how many eggs can three chickens lay in three days? Fibonacci 800 years ago in his book *Liber Abaci* (1202) did not have exactly this problem (as far as I could find), but he posed its equivalent. And most likely the problem came even earlier from the Arabs. So we can essentially claim Fibonacci (or the Arabs) as the father of the chicken and egg problem. Here are three of Fibonacci's actual problems ([1]).

1. Five horses eat 6 sestari of barley in 9 days; it is sought by the same rule how many days will it take ten horses to eat 16 sestari.
2. A certain king sent indeed 30 men to plant trees in a certain plantation where they planted 1000 trees in 9 days, and it is sought how many days it will take for 36 men to plant 4400 trees.
3. Five men eat 4 modia of corn in one month, namely in 30 days. Whence another 7 men seek to know by the same rule how many modia will suffice for the same 30 days.

By modern standards these problems all involve simple arithmetic to solve. But there are actually some subtleties in mapping the mathematical model to the situation, in which fractions, proportions, ratios, and "direct variation" get swirled into the mix—naturally causing some confusion.

Solution

The simple solutions are given in the following. We also supply the solution Fibonacci (Leonardo of Pisa) gave to the first problem.

1. **Problem** ([1] p.206)**:** "Five horses eat 6 sestari of barley in 9 days; it is sought by the same rule how many days will it take ten horses to eat 16 sestari."

Modern Solution:

b = sestari of barley

h = number of horses

d = number of days

r = rate of barley eaten per horse per day

Model: $b = r\,h\,d$

Given: 6 sestari of barley = r x 5 horses x 9 days. Therefore, r = 6/45 = 2/15 sestari per horse per day. Then 16 = r 10 d = (2/15) 10 d = (4/3) d or d = (3/4) 16 = 12 days.

Leonardo Solution ([1] p.206): You write the 5 on the upper line for horses, and afterwards the 6 for the barley, and the 9 for the days; and below the 5 you put the 10 horses, and you put 16 sestari below the 6, and you multiply the 5 by the 16 and by the 9; there will be 720 that you divide by the 6 and the 10; the quotient will be 12 days; or in another way, if 5 horses eat 6 sestari in 9 days, then ten horses eat double 6 sestari in 9 days, as 10 horses are double 5 horses. Again because 10 horses eat 12 sestari in 9 days, then they eat 16 sestari in 12 days, which results from multiplying the 16 by the 9 and dividing by the 12. We can show in this problem 18 combinations of proportions ….

days	barley	horses
9	6	5
$\boxed{12}$	16	10

Commentary:

We can translate Leonardo's words into the arithmetic steps we used in the solution. We began with the relation b = r h d and were given specific values for b, h, and d, namely, $b_0 = 6$ sestari of barley, $h_0 = 5$ horses, and $d_0 = 9$ days. From that we computed the (constant) rate as $r = b_0 / h_0 d_0 = 6/(5 \cdot 9)$. Then we considered how many days d would it take h = 10 horses to eat b = 16 sestari. So that would be solving for d in the relation b = r h d, or

$$d = \frac{b}{rh} = \frac{h_0 d_0 b}{b_0 h} = \frac{5 \cdot 9 \cdot 16}{6 \cdot 10} = 12$$

and so we see the calculation described in blue letters above. Leonardo solved the problem in words where we would use symbols for the variables and operations. (Leonardo knew how to multiply fractions and that b/a is the reciprocal of a/b, but his time was some 400 years before the development of a full symbolic algebra.)

In addition, it seems Leonardo was imagining the problem in terms of proportions rather than direct arithmetic manipulations solving for an unknown. The heavy emphasis on proportions is a hold-over from the Greeks. We shall address this in a moment.

2. **Problem** ([1] p.210)**: "**A certain king sent indeed 30 men to plant trees in a certain plantation where they planted 1000 trees in 9 days, and it is sought how many days it will take for 36 men to plant 4400 trees."

Modern Solution:

t = number of trees

m = number of men

d = number of days

r = rate of trees planted per man per day

Model: t = r m d

Given: 1000 trees = r x 30 men x 9 days. Therefore r = 1000/270 = 100/27 trees per man per day. Then 4400 = r 36 d = (100/27) 36 d = (400/3) d or d = (3/400) 4400 = 33 days.

3. **Problem** ([1] p.211)**: "**Five men eat 4 modia of corn in one month, namely in 30 days. Whence another 7 men seek to know by the same rule how many modia will suffice for the same 30 days."

Modern Solution:

c = amount of corn

m = number of men

d = number of days

r = rate of corn eaten per man per day

Model: c = r m d

Given: 4 modia of corn = r x 5 men x 30 days. Therefore r = 4/150 = 2/75 modia of corn per man per day. Then c = r x 7 men x 30 days = (2/75) 210 = 28/5 = $5\ ^3/_5$ modia of corn.

Mathematical Model

Where did we get the mathematical model for the problems, for example, $b = r\,hd$, where r is a constant, in Problem 1? Leonardo's second solution in Problem 1 reveals the type of relationship that is being considered:

> or in another way, if 5 horses eat 6 sestari in 9 days, then ten horses eat double 6 sestari in 9 days, as 10 horses are double 5 horses. Again because 10 horses eat 12 sestari in 9 days, then they eat 16 sestari in 12 days, which results from multiplying the 16 by the 9 and dividing by the 12.

This phraseology suggests a proportional relationship in each of the variables. But what does that *really* mean?

Proportions and Fellow Travelers

James Tanton in his excellent essay on Proportion and Ratio ([2]) tries to sort through the confusion that often attends these ideas:

> Ever since the release of the Common Core State Standards I've been afraid to admit that I don't understand the subtleties of "ratio and proportion," at least, I was under the impression that I don't. … I thought the Common Core was using the word *proportion*, a word that I actually don't understand. It doesn't. The Common Core repeatedly uses the phrase *proportional relationship* instead, which emphasizes connection between two quantities, as it should. It rightly removes the hazy use of *proportion* as a stand-alone word.

So Tanton defines a proportional relationship as follows:

> Two quantities appearing in a scenario are said to be in a *proportional relationship*, or just *proportional*, if doubling the amount of one quantity forces the amount of the other to double as well, or tripling the amount of one quantity forces the amount of the other to also triple, or halving the amount of one forces the amount of the other to also halve. And so on.
>
> That is, two quantities are proportional if changing the amount of one of the quantities by some factor forces the amount of the other to change by that same factor too.

Deviating from Tanton's more concrete and elementary approach, I would like to couch the subject in terms of functional equations. Recall for our purposes that a function is a rule, designated by f say, which assigns to each element x of a set A of entities one and only one element y from a second set B of entities. We write this $y = f(x)$ or $f\colon x \to y$. Here we shall assume both sets A and B are the set of all real numbers. So for example, $y = f(x) = 2x^2 - 4x$ is a real-valued function of a real variable.

In terms of functions, Tanton's definition of a proportional relationship (highlighted in yellow above) is represented by a function f on the real numbers which satisfies

$$f(tx) = tf(x) \text{ or } f\colon tx \to ty \quad \text{for all real numbers } t \text{ and } x \text{ (where } y = f(x)).$$

From this functional definition can we characterize f, that is, completely describe its form? Notice that $f(0\cdot x) = 0\cdot f(x)$ means $f(0) = 0$. Furthermore,

$$f(x) = f(x\cdot 1) = x\cdot f(1) \;\Rightarrow\; y = f(x) = kx \quad \text{for constant } k = f(1)$$

And there we have it. Two quantities x and y are proportional if one is a constant multiple of the other. The constant k is often called a *scale factor* or a rate. In further terminology, y is said to *vary directly* with x.

For me the primary application of the idea of proportionality is in geometry where one figure is proportional (similar) to another if the length of each edge in one figure is a constant multiple of the length of the corresponding edge in the other figure. One figure is a "scaled" version of the other.

Ratios. Now where the plot thickens is algebraically

$$y = kx \text{ is equivalent to } y/x = k.$$

Here the symbology y/x represents implied division.

We have traditionally called such symbology a "ratio" and it is intimately involved with the Greek idea of proportions. Since the proportional property means ty/tx = k also, we have that two quantities y and x are proportionally related if y/x = ty/tx for all t. The Greeks phrased this ([4] Book V Def.6) "Let magnitudes which have the same ratio be called *proportional.*" and eventually the notation y : x :: ty : tx developed to capture this idea (without indicating division in any way). Verbally, one says "y is to x as ty is to tx." Euclid then went on to establish a number of properties of proportional magnitudes that we would obtain today by simple symbolic algebra manipulations.

This ratio symbology also represents something else, a number in fact, which we call a fraction, where y is called the "numerator" and x is called the "denominator", but this was unknown to the Greeks.[1] In the parlance of fractions, the numerator and denominator are whole numbers (eventually including negative integers and zero, the last in the numerator only). The definition and mathematical manipulation of general fractions came to us from the Indians[2] via the Arabs through Fibonacci himself in his *Liber Abaci*.[3] Again, James Tanton ([3]) gives a nice, concrete approach to the meaning and manipulation of fractions. The simplicity of "modern" algebraic manipulations in place of proportions can be illustrated in the proportional expression ty/tx = y/x since ty/tx = (t/t)(y/x) = 1·y/x = y/x. And so without resorting to proportions, we can "reduce" the fraction 6/45 via (3·2)/(3·15) to (3/3)(2/15) = 2/15.

So there is a lot going on here. The symbolic algebra we have become used to hides multiple perspectives that all arrive at the same notation: *a*/*b* can represent a fraction, a proportional

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1 What we call "rational numbers" or the ratio of whole numbers, the Greeks viewed as "commensurable" numbers. That is, first of all the Greeks thought of "numbers" as only counting something, measuring the multiplicity of things. Two entities were commensurable if a common unit of measure could be found such that the two entities were each an integral number of these units. Thus the lengths of two sticks would be commensurate if, say, one was 8 inches long and the other 5 inches long or 11 centimeters versus 6 centimeters. If no such common unit of measure could be found, the two entities were considered "incommensurate". And so the hypotenuse of an isosceles right triangle is incommensurate with the legs, that is, there is no common unit of measure such that the hypotenuse and legs have an integral number of these units. Rather than try to assign some sort of new "number" to the hypotenuse, the Greeks left it as a geometric construction. And so any problem that entailed what we call irrational numbers would be handled by the Greeks through geometric constructions. Solving an equation for a value x would be handled by constructing in a figure an edge of length x. Still, there was a type of mathematical manipulation of incommensurate quantities, but it was achieved through the Greek ideas of proportions. (See e.g., [4] Book V Defs 1-6; [5] pp.1-18; [6])

2 Starting in the 2nd millennium BC ([7] pp.185-203)

3 Other civilizations used "fractions", but in a limited sense and not always with the idea of numerators and denominators. (See any book on the history of mathematics.) That is, measures of quantities would be subdivided into units like for length (yards, feet, and inches), time (days, hours, minutes, and seconds), or especially money (pounds, shillings, pence). Note that these subdivisions are not decimal and involve different amounts like 3 feet in a yard and 12 inches in a foot, or 24 hours in a day but 60 minutes in an hour, etc. Still they could be manipulated in an abacus where each column represented the amounts of one of the subdividing units. None of these operations involved the arithmetic rules of numerator/denominator fractions. The Babylonians seemed to understand the idea of implied division in a ratio by converting the denominator into its reciprocal and multiplying. They built tables of reciprocals and then multiplied within their sexagesimal place system. I haven't studied the details, so I can't shed further light on the Babylonians, but their number system finally infiltrated Greek mathematics via Ptolemy (c.150 AD) ([5] p.16).

relationship between *a* and *b*, or a division of *a* by *b*. This "redundancy" makes the expressions look simple, but it hides a lot of ideas that all converge to the same place from vastly different historical origins. Instead of producing harmony, this historical baggage can sometimes create dissonance, which can manifest itself in confusion.[4] Alternatively, the abstraction process in mathematical development can reveal an unsuspected commonality that is inherent in multiple, seemingly different settings.

Back to the Model

The models in the *Liber Abaci* problems involve two variables. As Leonardo wrote in Problem 1, the amount of barley b is proportional not only to the number of horses h, but also to the number of days d. From a functional point of view, we are interested in a real-valued function $z = f(x, y)$ of two real variables x and y such that

$$f(tx, y) = tf(x, y) = f(x, ty) \text{ for all real } t, x, y$$

So $f: (tx, y) \rightarrow tz$ and $f: (x, ty) \rightarrow tz$. And so $f(tx, ty) = t^2 f(x, y) = t^2 z$. As before $0 = f(0, y) = f(x, 0) = f(0, 0)$. And

$$z = f(x, y) = k\,xy \quad \text{for constant } k = f(1, 1)$$

So this is where our model comes from for the problems. The constant k represents the rate or "how much of z for one x and for one y," phrased "the amount of z per x per y."

Postscript

Can you solve the chicken and the egg problem now?

It is easy to show the proportionality property, $f(tx) = tf(x)$ for all real t and x, implies the linearity property, $f(x + y) = f(x) + f(y)$ for all real x and y. ($f(x + y) = (x + y)f(1) = xf(1) + yf(1) = f(x) + f(y)$). The converse, where f is assumed continuous, is a standard problem in calculus courses. That is, if f is continuous, show $f(x + y) = f(x) + f(y)$ for all real x and y implies there is a constant k such that $f(x) = kx$ for all real x.

In other settings these ideas are very powerful and come to dominate the subject. If f is a function between vector spaces V and W over the scalar field K, then f is called a *linear transformation* if for all scalars a and b and vectors **v** and **w**,

$$f(a\mathbf{v} + b\mathbf{w}) = a\,f(\mathbf{v}) + b\,f(\mathbf{w}).$$

Instead of multiplication by a constant (i.e., scalar) f is characterized by multiplication of a vector by a (constant) matrix. Notice that if the reals R are thought of as a vector space over themselves, then the dimension is 1 and the matrix is 1×1, or equivalently a scalar. More convergence of ideas—or, perhaps more accurately, expansion.

References

[1] Sigler, Laurence E., tr. *Liber Abaci*, Springer, 2002

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[4] Soapbox: I dislike using classical proportions in any setting other than similar geometric figures, that is, non-geometric settings where one is using the relationship "a is to b as c is to d" (a : b :: c : d), such in the modern "Problem 3. Proportional Relationships" cited by the usually perspicacious David Bressoud, whose historical approach to modern mathematics I greatly admire ([8]). I believe such a continued emphasis on proportions is an obsolete traditional holdover due to the dominance of Greek mathematics (Euclid's *Elements*) for over 2000 years up to the 17^{th} century. I prefer to go directly to the mathematical model, e.g. $z = r\,xy$, by emphasizing direct variation, and solve the problem with algebraic manipulations, though Tanton's idea of proportionality is implicit (the rate r is assumed constant).

[2] Tanton, James, “Tanton’s Take on Proportion and Ratio”, *Curriculum Essays*, December 2016 (http://www.jamestanton.com/wp-content/uploads/2012/03/Curriculum-Essay_December-2016_Ratio-and-Proportion.pdf)

[3] Tanton, James, *Fractions are Hard!* February 2016 (http://gdaymath.com/courses/fractions-are-hard/)

[4] Joyce, D.E., *Euclid’s Elements*, 1998 (http://aleph0.clarku.edu/~djoyce/java/elements/elements.html)

[5] Toeplitz, Otto, *The Calculus: A Genetic Approach*, University of Chicago Press, 1963 (from 1949 German version), reprint 2007 with Forward by David Bressoud.

[6] Grattan-Guinness, Ivor, “Numbers, Magnitudes, Ratios, and Proportions in Euclid’s Elements: How Did He Handle Them?”, *Historia Mathematica* 23 (1996), 355-375 Article No. 0038 (www.tau.ac.il/s-corry/teaching/toldotidownload/IGG.pdf)

[7] Datta, Bibhutibhusanl and Avadhesh Narayan Singh, *History of Hindu Mathematics, A Source Book Part* 1 (Numeral notation and arithmetic), Motilal Banarsidass; Lahore; 1935. pp.185-203 (https://archive.org/download/in.gov.ignca.9325/9325_text.pdf)

[8] Bressoud, David, “IJRUME: Measuring Readiness for Calculus,” *Launchings*, 1 November 2016 (http://launchings.blogspot.com/2016/11/ijrume-measuring-readiness-for-calculus.html)