Circular Rendezvous Mystery

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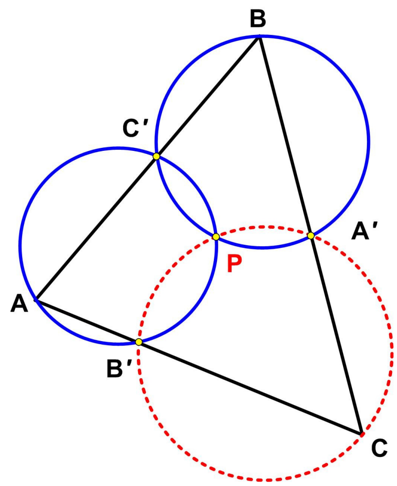
Here is yet another surprising result from Colin Hughes at *Maths Challenge*.[1]

Problem

It can be shown that a unique circle passes through three given points. In triangle ABC three points A', B', and C' lie on the edges opposite A, B, and C respectively. Given that the circle AB'C' intersects circle BA'C' inside the triangle at point P, prove that circle CA'B' will be concurrent with P.



My Solution

I approached the problem a bit differently. Rather than prove a circle CA'B' goes through P, I prove equivalently that a circle PA'B' goes through C (Figure 1). I have to admit it took me a while to arrive at the final version of my proof. My original approach had some complicated expressions using various angles, and then I realized I had not used my assumption that the circle went through P. Once I involved P, all the complications faded away and the result became clear.

From the original triangle we have angles A + B + C = 180º. Furthermore, from Figure 1 angles A = $\alpha/2$, B = $\beta/2$, and D = $\delta/2$. Then from Figure 2 we have

$$360^\circ = \alpha' + \beta' + \delta' = (360^\circ - \alpha)/2 + (360^\circ - \beta)/2 + (360^\circ - \delta)/2$$

$$= (180^\circ - A) + (180^\circ - B) + (180^\circ - D)$$

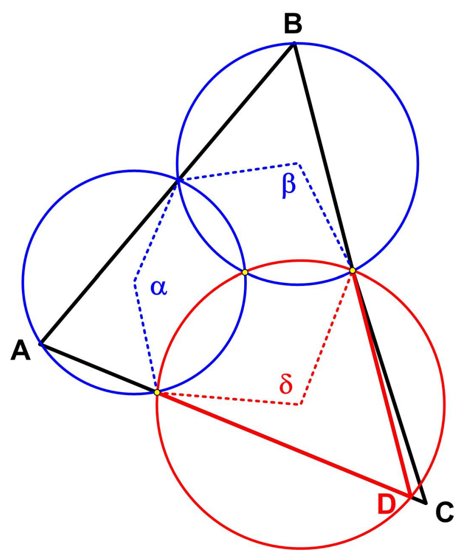


Figure 1

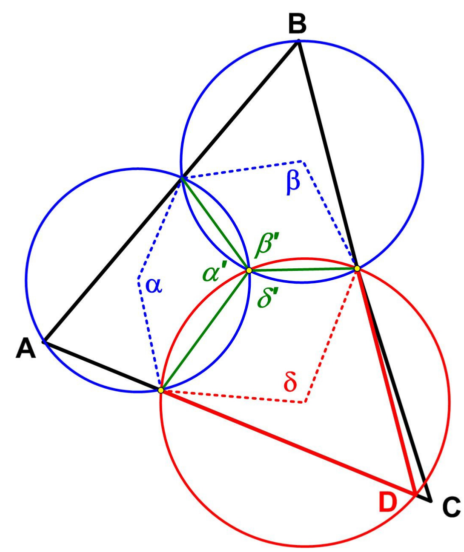


Figure 2

[1] "Concurrent Circles in a Triangle" Problem ID: 322 (14 Apr 2007) Difficulty: 3 Star at mathschallenge.net. "A three-star problem: a good knowledge of school mathematics and/or some aspects of proof will be required." (https://mathschallenge.net/problems/pdfs/mathschallenge_3_star.pdf)

Therefore

$$A + B + D = 180^o$$

and so angles

$$D = C$$

which means the circle PA’B’ must pass through the vertex C.

Maths Challenge Solution

Here is the Maths Challenge solution which is basically what I did, though worded differently.

Consider the following diagram (Figure 3). As AB'PC' is a cyclic quadrilateral[2]

angle A + angle B'PC' = 180 degrees.[3]

Similarly A'PC'B is a cyclic quadrilateral so

angle B + angle A'PC' = 180 degrees.[4]

Therefore

angle A + angle B = 360 – (angle B'PC' + angle A'PC')

= angle A'PB'.

However, in triangle ABC,

angle A + angle B = 180 – angle C.

Hence

angle A'PB' = 180 – angle C[5]

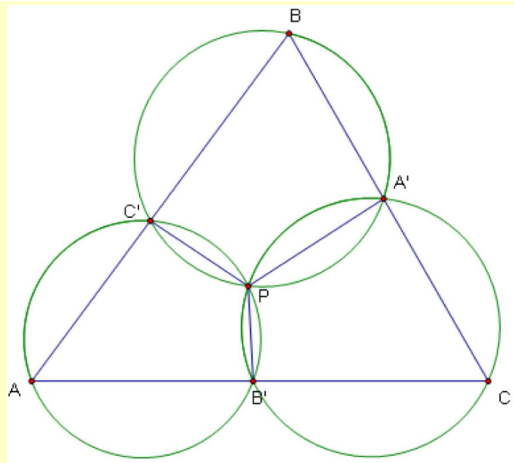


Figure 3

and we show that CA’PB’ is a cyclic quadrilateral and the circle passing through C, A’, and B’ is concurrent at P.

This result is known as Miquel’s theorem and remains true if the common point is outside the triangle…

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[2] JOS: (*Wikipedia*) “In Euclidean geometry, a cyclic quadrilateral or inscribed quadrilateral is a quadrilateral whose vertices all lie on a single circle.”

[3] JOS: This is my result: $\alpha' = 180 - A$.

[4] JOS: This is my result: $\beta' = 180 - B$.

[5] JOS: The rest of the proof is essentially what I did as well.