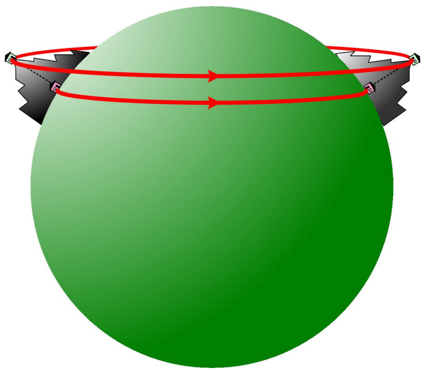
Mountain Houses Problem

5 August 2019

Jim Stevenson



It is always fascinating to look at problems from the past. This one, given by Thomas Whiting himself, is over 200 years old from Whiting's 1798 *Mathematical, Geometrical, and Philosophical Delights* ([1] p.6):

Question 2, *by T. W. from Davison's Repository*.

There are two houses, one at the top of a lofty mountain, and the other at the bottom; they are both in the latitude of 45°, and the inhabitants of the summit of the mountain, are carried by the earth's diurnal rotation, one mile an hour more than those at the foot.

Required the height of the mountain, supposing the earth a sphere, whose radius is 3982 miles.

Whiting's Solution

For clarity I have interpolated the notation from Figure 1 in blue text in the solution, since Whiting had a propensity to carry out the arithmetic immediately rather than represent it symbolically.

Solution by the Proposer.

Let P (Fig. 1) be the pole of the world, C the earth's centre, A the bottom of the mountain, B the top, AD a perpendicular let fall from the bottom of the mountain upon the earth's axis, and BE another perpendicular let fall from the top to ditto. By trig. As radius : 3982 :: cos lat. 45° : AD = 2815.69 [$R\cos\lambda$] and 2815.69 x 2 x 3.14159 [$2\pi R\cos\lambda$], &c. = 17691.54 = the circumference of the given parallel of latitude, or distance the inhabitant's at the foot of the mountain are carried in 24 hours. Then 17691.54 + 24 = 17715.54 [$24 + 2\pi R\cos\lambda = 2\pi(R + h)\cos\lambda$] the space gone over by the inhabitants at the top of the mountain. Hence as the lines AC and BC describe similar cones about CD and CE, we have this analogy, as 17691.54 : 3982 :: 17715.54 : 3987.45 = CB [$2\pi R\cos\lambda\ /\ R = (24 + 2\pi R\cos\lambda) / (R + h)$], hence 3987.45 – 3982 = 5.45 miles the height of the mountain.

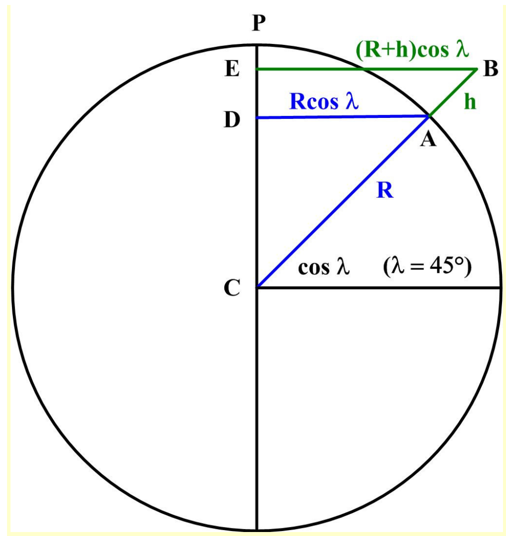


Figure 1 Question 2 Setup (Augmented)

My Solution

It turns out that we do not need to compute the radius R. That is, if we let v_1 be the speed of the house at the foot of the mountain, v_2 the speed of the house at the top, and latitude $\lambda = 45°$, then

$$v_1 = 2\pi R\cos\lambda / 24 \text{ mph}$$

$$v_2 = 2\pi(R+h)\cos\lambda / 24 \text{ mph}$$

So $v_2 - v_1 = 1$ mph means

$$1 = (2\pi\cos\lambda / 24)[(R + h) - R]$$

or

$$\boxed{h = 12 / (\pi\cos\lambda) \approx 3.82 / \cos\lambda}$$

For $\lambda = 45°$, $\cos\lambda = 1/\sqrt{2}$, so $h = 5.4$ miles. Notice that h does not depend on the radius of the earth, but only the latitude.

Apparently they really liked to calculate a lot of arithmetic back in the late 1700s. I used an excerpt from Whiting's Preface for the epigraph on my About page. It is amazing how apropos it was, even after 200 years.

The following is an image of the original problem (Figure 2).

6 *Mathematical, Geometrical, and Philoſophical Delights.*

Queſtion 2, *by T. W. from Daviſon's Repoſitory.*

There are two houſes, one at the top of a lofty mountain, and the other at the bottom; they are both in the latitude of 45°, and the inhabitants of the ſummit of the mountain, are carried by the earth's diurnal rotation, one mile an hour more than thoſe at the foot.

Required the height of the mountain, ſuppoſing the earth a ſphere, whoſe radius is 3982 miles.

Solution by the Propoſer.

Let P *(Fig. 1)* be the pole of the world, C the earth's centre, A the bottom of the mountain, B the top, AD a perpendicular let fall from the bottom of the mountain upon the earth's axis, and BE another perpendicular let fall from the top to ditto. By trig. As radius : 3982 :: coſ. lat. 45° : AD = 2815.69 and 2815.69 × 2 × 3.14159, &c. = 17691.54 = the circumference of the given parallel of latitude, or diſtance the inhabitants at the foot of the mountain are carried in 24 hours. Then 17691.54 + 24 = 17715.54 the ſpace gone over by the inhabitants at the top of the mountain. Hence as the lines AC and BC deſcribe ſimilar cones about CD and CE, we have this analogy, as 17691.54 : 3982 :: 17715.54 : 3987.45 = CB, hence 3987.45 − 3982 = 5.45 miles the height of the mountain.

Figure 2 Image of Original Problem

References

[1] Whiting, Thomas, *Mathematical, Geometrical, and Philosophical Delights, Containing Essays, Problems, Solutions, Theorems, &c. Selected from an Extensive Correspondence*, London, 1798 (https://archive.org/details/mathematicalgeo00unkngoog/page/n7, retrieved 4/27/2019)