Maximum Product

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This 2007 four-star problem from Colin Hughes at *Maths Challenge*[1] is definitely a bit challenging.

Problem

For any positive integer, k, let $S_k = \{x_1, x_2, \dots , x_n\}$ be the set of [non-negative] real numbers for which

$$x_1 + x_2 + \dots + x_n = k$$

and $P = x_1\, x_2 \dots x_n$ is maximised. For example, when $k = 10$, the set {2, 3, 5} would give P = 30 and the set {2.2, 2.4, 2.5, 2.9} would give P = 38.25. In fact, S_{10} = {2.5, 2.5, 2.5, 2.5}, for which P = 39.0625.

Prove that P is maximised when all the elements of S are equal in value and rational.

I took a different approach from Maths Challenge, but for me, it did not rely on remembering a somewhat obscure formula. (I don't remember formulas well at my age—only procedures, processes, or proofs, which is ironic, since at a younger age it was just the opposite.) It is also clear from the Maths Challenge solution that the numbers were assumed to be non-negative.

My Solution

I saw the problem as a constrained optimization problem, which typically can be solved with Lagrange Multipliers (See for example [1] pp.859-861). We are trying to maximize the function $P = f$ where

$$f(x_1, x_2, \dots , x_n) = x_1\, x_2 \dots x_n \tag{1}$$

subject to the constraint

$$g(x_1, x_2, \dots , x_n) = x_1 + x_2 + \dots + x_n - k = 0. \tag{2}$$

In terms of the n-vector $\mathbf{x} = (x_1, x_2, \dots , x_n)$, we are trying to maximize $f(\mathbf{x})$ where $\mathbf{x}$ is constrained to satisfy $g(\mathbf{x}) = 0$.

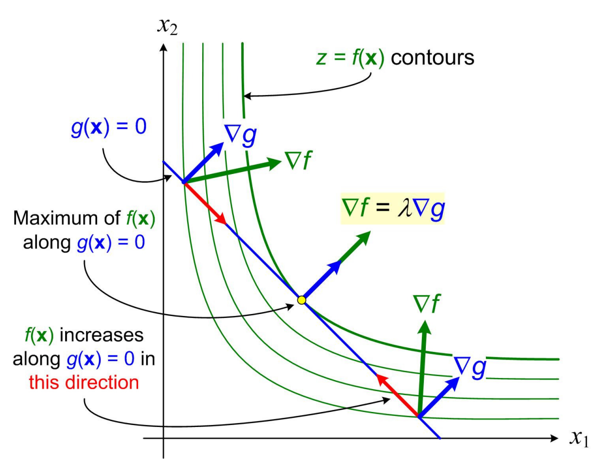

Figure 1 Lagrange Multiplier Idea

Figure 1 shows the situation (for n = 2) where we have represented f and g by their constant contours (like the height contours on a relief map). The gradient vectors ∇f and ∇g of these contours are perpendicular to them (perpendicular to their tangents) at each point and represent the direction of maximally increasing values. The red arrows in the figure represent the direction one could move along the contour $g(\mathbf{x}) = 0$ to increase the value of f ("hill climbing"). These directions of

[1] "Maximum Product" Problem ID: 335 (19 Nov 2007) Difficulty: 4 Star at mathschallenge.net. "A four-star problem: A comprehensive knowledge of school mathematics and advanced mathematical tools will be required." (https://mathschallenge.net/problems/pdfs/mathschallenge_4_star.pdf)

motion converge on the "highest" point or maximum for f along $g(\mathbf{x}) = 0$. At that point the gradient vectors of f and g are parallel, that is, $\nabla f = \lambda \nabla g$ for some scalar λ. λ is called a Lagrange multiplier.

(The rigorous proof of this idea is left to an advanced calculus course. Technically the Lagrange multiplier method provides an extreme point where the function f may be either a maximum, minimum, or neither (saddle point). There are expressions involving second partials to sort this out, but usually it is clear from the problem statement what the case might be.)

Let $\mathbf{e}_1 = (1, 0, \ldots, 0)$, $\mathbf{e}_2 = (0, 1, \ldots, 0)$, …, $\mathbf{e}_n = (0, 0, \ldots, 1)$ represent the unit basis vectors for $\mathbf{R}^n$. Then

$$\nabla f = \frac{\partial f}{\partial x_1}\mathbf{e}_1 + \ldots + \frac{\partial f}{\partial x_n}\mathbf{e}_n = x_2 x_3 \ldots x_n \mathbf{e}_1 + \ldots + x_1 x_2 \ldots x_{n-1}\mathbf{e}_n$$

and

$$\nabla g = \frac{\partial g}{\partial x_1}\mathbf{e}_1 + \ldots + \frac{\partial g}{\partial x_n}\mathbf{e}_n = \mathbf{e}_1 + \ldots + \mathbf{e}_n$$

Note that in each expression for $\partial f/\partial x_k$ only x_k is missing from the product in $P = f$. Also note that all the gradients have non-negative coefficients, which would imply heading toward a maximum.

So at the maximum point, $\nabla f = \lambda \nabla g$ implies that for every k, $x_k \lambda = P$, the entire product. Therefore,

$$x_1 = x_2 = \ldots = x_n = P/\lambda$$

Let x_0 represent the common value for all the xs. From equation (2) we have

$$k = x_1 + x_2 + \ldots + x_n = n\, x_0$$

so

$$x_0 = k/n$$

Therefore, the product $P = x_1\, x_2 \ldots x_n$ is maximal when all the xs are the same and rational. (Note in the example given in the problem statement, $k = 10$ and $n = 4$, so the common value is 10/4 = 2.5.)

Maths Challenge Solution

This proof will make use of the AM-GM inequality ([2]), which states that for any set of [non-negative] real numbers their arithmetic mean is greater than or equal to their geometric mean.

$$(x_1 + x_2 + \ldots + x_n)/n \geq (x_1\, x_2 \ldots x_n)^{1/n} \qquad (1)$$

In particular, equality is given if and only if $x_1 = x_2 = \ldots = x_n$.

Now [by assumption on S_k]

$$k/n = (x_1 + x_2 + \ldots + x_n)/n \geq (x_1\, x_2 \ldots x_n)^{1/n}$$

so

$$(k/n)^n \geq x_1\, x_2 \ldots x_n = P$$

[As $P \leq (k/n)^n$, by the equality condition of the AM-GM inequality we have that P will be maximised ($P = (k/n)^n$) when the terms are all equal. Let x_0 be that common value, that is, $x_0 = x_1 = x_2 = \ldots = x_n$. Then $(k/n)^n = P = (x_0)^n$ means the common value $x_0 = k/n$, a rational number.]

I actually reworded the original (blue text) and deleted the Maths Challenge proof that the common value for the xs was rational, which proceeded by maximizing (the already maximized) $P = (k/n)^n$ as a function of n, for some reason. Since k and n are given fixed, I did not follow his argument or the reason for it. I may be missing something, but what I wrote seems sufficient.

Certainly the Maths Challenge solution is shorter, but all the work is hidden in the AM-GM inequality, which I have trouble remembering—and that includes the proof (maybe by mathematical induction?). Lagrange multipliers are such a standard and ubiquitous method for constrained optimization that I naturally thought of them first. And the gradient computations are trivial in this case. Also, I naturally love the geometric flavor of Lagrange multipliers which offer a visual context for the problem (reflected accurately in Figure 1 for the case $n = 2$).

References

[1] Thomas Jr., George B. (late), Maurice D. Weir, Joel R. Hass, *Thomas' Calculus: Early Transcendentals* 13th Edition, Pearson, 1200 pp, 2014

[2] "Inequality of arithmetic and geometric means," *Wikipedia* (https://en.wikipedia.org/wiki/Inequality_of_arithmetic_and_geometric_means, retrieved 8/3/2019