Impossible Car Riddle

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This is another intriguing problem from Presh Talwalkar.

(https://mindyourdecisions.com/blog/2019/07/29/a-seemingly-impossible-car-riddle/)

A Seemingly Impossible Car Riddle

Posted July 29, 2019 By Presh Talwalkar.

A car travels 75 miles per hour (mph) downhill, 60 mph on flat roads, and 50 mph uphill. It takes 3 hours to go from town A to B, and it takes 3 hours and 30 minutes for return journey by the same route. What is the distance in miles between towns A and B?

A similar (and perhaps easier) problem was asked in a US Math Olympiad qualifying test (2017 AMC 10A, problem 9).

My Solution

Figure 1 shows the problem statement where for the trip from A to B we have collected all the down-slopes into one total distance d_1, all the up-slopes in one total distance d_3, and all the horizontal sections in one total distance d_2. Thus the total distance between A and B is

$$D = d_1 + d_2 + d_3$$

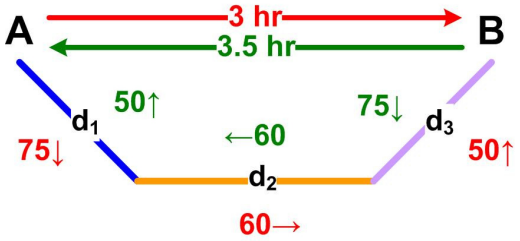

Figure 1 Problem Statement

We have also labeled the distances with the speeds in mph corresponding to whether the car is driving up a slope, down a slope, or horizontally. And we have shown the time to drive from A to B is 3 hours and from B to A is 3.5 hours. With these speeds, distances, and times we get the following two equations,

for A to B: $$3 = d_1/75 + d_2/60 + d_3/50 \quad (1)$$

and for B to A: $$3.5 = d_1/50 + d_2/60 + d_3/75 \quad (2)$$

Subtracting the first from the second yields

$$½ = (1/50 - 1/75)(d_1 - d_3)$$

or, multiplying both sides by 150,

$$75 = d_1 - d_3$$

This result is shown in Figure 2. This distance was driven in 1 hour on the trip from A to B (75 mi/75 mph = 1 hr) and 1.5 hours on the return trip from B to A (75 mi/50 mph = 1.5 hr). This section takes care of the discrepancy in time between the trip from A to B and the

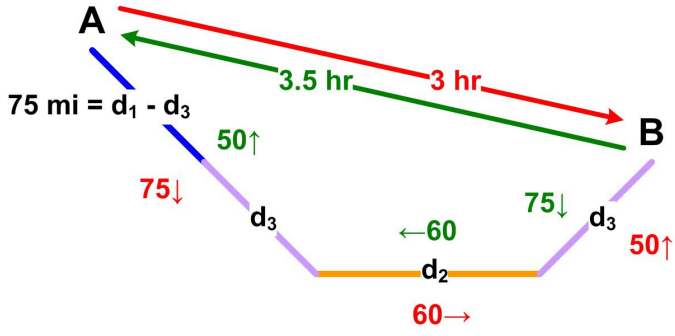

Figure 2 First Step in Solution

return. Therefore the remaining parts of the trips both take 2 hours. So we get the equation

$$2 = d_3/50 + d_2/60 + d_3/75 = d_3/30 + d_2/60 = (2d_3 + d_2)/60$$

or $$120 = 2d_3 + d_2$$

Thus $$D = d_1 + d_2 + d_3 = (d_1 - d_2) + 2d_2 + d_3 = 75 + 120 = 195 \text{ miles.}$$

Talwalkar Solution

Talwalkar set up the problem the same way, but he added the two equations (1) and (2) rather than subtracted. I should have seen the symmetries and done the same, but I didn't. In any case, adding the equations and noting that 1/50 + 1/75 = 1/30 as we saw above yields

$$6.5 = d_1/30 + d_2/30 + d_3/30 = D/30$$

(This nice clean result was certainly not evident to me at first, which is why I subtracted, trying to eliminate one of the variables, namely d_2.) Thus

$$D = 195 \text{ miles}.$$

Talwalkar also provided two sources:

1. Nick's mathematical puzzle, number 75, http://www.qbyte.org/puzzles/p075s.html
2. AoPS 2017 AMC 10A, problem 9 (the AMC is a qualifying test to the US Olympiad) https://artofproblemsolving.com/wiki/index.php/2017_AMC_10A_Problems/Problem_9