Magic Pythagorean Circle

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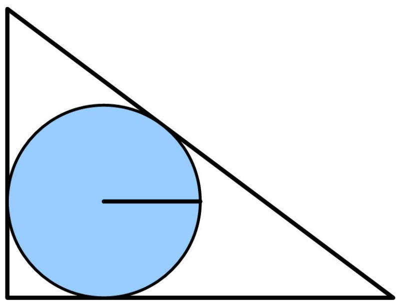


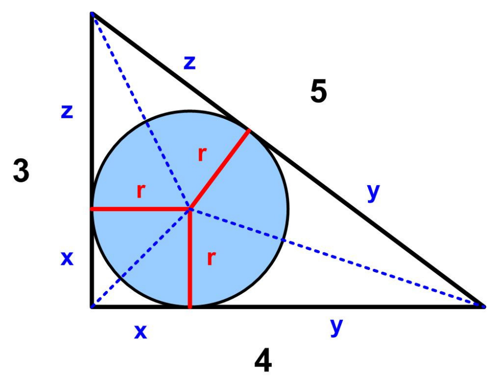
This statement showed up recently at Futility Closet[1] and I found it to be another one of those magical results that seemed so surprising. I don't recall ever seeing this before.

The radius of a circle inscribed in a 3-4-5 triangle is 1.

(In fact, **the inradius of *any* Pythagorean triangle is an integer.**)

(A Pythagorean triangle is a right triangle whose sides form a Pythagorean triple.) Futility Closet left these remarkable statements unproven, so naturally I felt I had to provide a proof.

Radius of Circle in 3-4-5 Triangle is 1



Label the circle and triangle as shown in the figure at right. By multiple applications of the Pythagorean theorem the three blue dashed lines carve up the original triangle into three pairs of congruent triangles, justifying the common lengths indicated by the variables x, y, and z.

Then we have the following equations.

$$x + z = 3$$

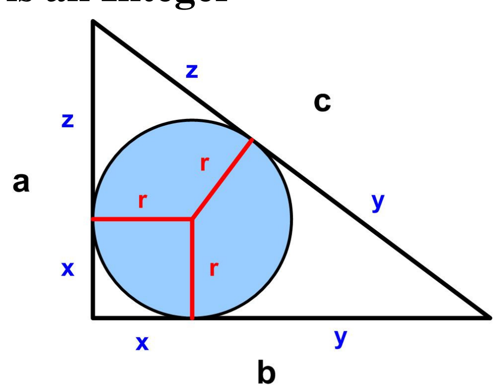
$$x + y = 4$$

$$y + z = 5$$

Given the perpendicularity of the radii of the circle with the tangential legs of the right triangle, the radii and the sections of the legs labeled x form a square. Therefore we have $x = r$. Plugging this into the three equations yields.

$$7 = 2r + (y + z) = 2r + 5 \;\Rightarrow\; r = 1$$

Inradius of Any Pythagorean Triangle is an Integer



We can essentially apply the same argument to the general case where the legs of the right triangle are labeled a and b and the hypotenuse labeled c. Then we have the three equations

$$x + z = a$$

$$x + y = b$$

$$y + z = c$$

By the same argument as before, $x = r$. Solving for r, we get

[1] "Neat" *Futility Closet*, 9 July 2019 (https://www.futilitycloset.com/2019/07/09/neat-4/)

$$r = \frac{a+b-c}{2}$$

Therefore, r will be an integer if $a + b - c$ is even.

Claim. $a + b - c$ is even

This is quite remarkable, but unfortunately we have to proceed by cases.

Not all sides are even. First, note that the sides cannot all be even, since a Pythagorean triple is assumed to have no common factors, that is, we have scaled the right triangle to its smallest size. (Actually, I discovered this "smallest" triple is called a primitive Pythagorean triple. So we shall only considered those. If the claim is true for primitive triples, then it is true for all multiples of them.)

Two sides cannot be even. If two sides are even, then the third must also be even. Using the Pythagorean relationship $c^2 = a^2 + b^2$ means if a and b are even, then so is c. Similarly, if a and c are even, then subtracting a^2 from c^2 we have b even. And the same if b and c are even.

Not all sides are odd. Suppose they were. Then $a = 2k + 1$, $b = 2m + 1$, and $c = 2n + 1$ for some integers k, m, and n. But

$$a^2 + b^2 = (2k+1)^2 + (2m+1)^2 = 2 \times (\text{terms}) + 1 + 2 \times (\text{terms}) + 1 = 2 \times (\text{terms})$$

which is even. And

$$c^2 = (2n+1)^2 = 2 \times (\text{terms}) + 1$$

is odd—a contradiction.

Therefore, **two sides are odd and one is even**. Which means $\boldsymbol{a + b - c}$ **is even**.

And so r is an integer!

The fact that the sides form a Pythagorean triple (have integral values) is enough for the inscribed circle to have an integral radius. Marvelous.

Just as an example, consider a 5-12-13 triangle

$$r = (5 + 12 - 13)/2 = 4/2 = 2$$
