Cube Slice Angle Problem

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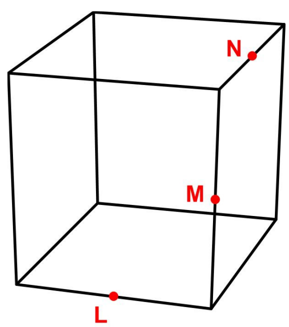
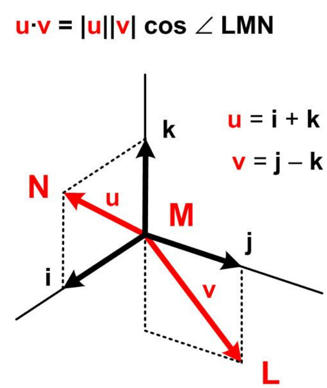


This is from the UKMT Senior Challenge of 2004.

L, M, and N are midpoints of a skeleton cube, as shown. What is the value of angle LMN?

A 90° B 105° C 120° D 135° E 150°

My Solution



My approach uses vectors and the dot product to obtain the angle between them. Run a vector **u** from the point M to the point N and a vector **v** from M to L. Locate the origin of the three dimensional coordinate system at M with **u** lying in the xz-plane and **v** lying in the yz-plane as shown in the figure at right. L, M, and N being midpoints means that L and N are the same distance from each of their corresponding coordinate axes so that the vectors take the form

$$\mathbf{u} = \mathbf{i} + \mathbf{k}$$

$$\mathbf{v} = \mathbf{j} - \mathbf{k}$$

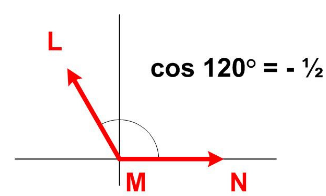
Then the dot product yields

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \angle LMN$$

$$0 + 0 - 1 = \sqrt{2}\,\sqrt{2}\, \cos \angle LMN$$

so $\cos \angle LMN = -1/2$

and $\angle LMN = 120°$ (Answer C)



UKMT Solutions

Solution 1. Let the side of the cube be of length 2. Then

$$LM = MN = \sqrt{1^2+1^2} = \sqrt{2} \text{ and } LM = \sqrt{1^2+1^2+1^2} = \sqrt{6}$$

So *LMN* is an isosceles triangle with sides $\sqrt{2}$, $\sqrt{2}$, $\sqrt{6}$. Thus

$$\cos \angle NLM = \frac{\sqrt{6}/2}{\sqrt{2}} = \frac{\sqrt{3}}{2}$$

Hence $\angle NLM = 30° = \angle MNL$. So $\angle LMN = 120°$.

Solution 2. Alternatively, it may be shown that L, M and N, together with the midpoints of three other edges of the cube, are the vertices of a regular hexagon. So $\angle LMN$ may be shown to be 120°.