Alberti's Perspective Construction

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I was mitigating the wait in doctors' offices (and trying to ignore the ubiquitous, annoying television) by dipping into David Wootton's *The Invention of Science* (2015) ([1]), in particular, his Chapter 5 with the provocative title "The Mathematization of the World" covering the 15th and 16th centuries. He discusses the arrival of bookkeeping and then the invention of perspective in painting, but seems to lose his way in the minutiae without really explaining how this related to his chapter title. In any case, the topic revived a historical interest in the subject which I had considered mathematically in my post "The Perspective Map".

Wootton's history of perspective focused mainly on Filippo Brunelleschi (1377 – 1446) of *Brunelleschi's Dome* ([2]) fame and Leon Battista Alberti (1404 – 1472) and his tome *De Pictura* (*On Painting*) (1435-6) ([3]), which contained the first mathematical presentation of perspective. It is not entirely clear what the distinctions were between Brunelleschi's and Alberti's contributions, but as noted by the translator of *On Painting*, John Spencer, "Geometry does not enter into Brunelleschi's construction, for it relies solely on sightings." ([3] Book 1, Note 48 p.113) Alberti introduces geometry via similar triangles to quantify the sizes of the figures and objects in a painting as they are transformed by the perspective map. Spencer further states "The theory outlined here as a source of Alberti's construction does not make use of trigonometry which had not yet been invented in his time."([3] Book 1, Note 48 p.114). This is a bit extreme. Trigonometry was known from Hellenistic times over some 1500 years earlier culminating in Ptolemy's (AD c.100 – c.170) *Almagest*, but was mainly focused on *spherical* trigonometry for astronomy, rather than *plane* trigonometry for surveying and the like. The explicit development of plane trigonometry did revive in the late 15th century, after Alberti.

I am only going to briefly summarize how Alberti explained things in his book; others have described it in more detail. My main goal is to see how much of the perspective map's properties I can glean from Alberti's simple construction and explanations without resorting to the math (trigonometry) I used before. In other words, how did Alberti do it without trig? I will use lots of diagrams.

Alberti's Fundamental Constructs

Alberti introduces some constructs that support his perspective computations, namely the *visual pyramid* and *cross-section*. The following quotes are from Spencer's translation of Book 1 of Alberti's *On Painting* ([3]) (I have omitted the footnote references unless needed.):

> The [visual] pyramid is a figure of a body from whose base straight lines are drawn upward, terminating in a single point. The base of this pyramid is a plane which is seen. The sides of the pyramid are those rays which I have called extrinsic. The cuspid, that is the point of the pyramid, is located within the eye where the angle of the quantity is. [pp.47-48] … [JOS: see Figure 1]

> Now, since in a single glance not only one plane but several are seen, we will investigate in what way many conjoined [planes] are seen. [p.51] … Where this is a single plane, either a wall or a panel on which the painter attempts to depict several planes comprised in the visual pyramid, it would be useful to cut through this pyramid in some definite place, so the painter would be able to express in painting similar outlines and colours with his lines. He who looks at a picture, done as I have described [above], will see a certain cross-section of a visual pyramid, artificially represented with lines and colours on a certain plane according to a given distance, centre and lights. Now, since we have said that the picture is a cross-section of the pyramid we ought to

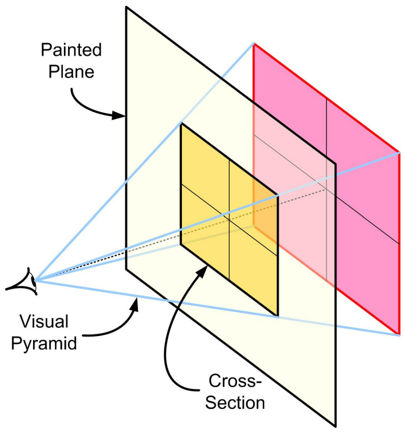


Figure 1 Alberti's Visual Pyramid with Cross-Section

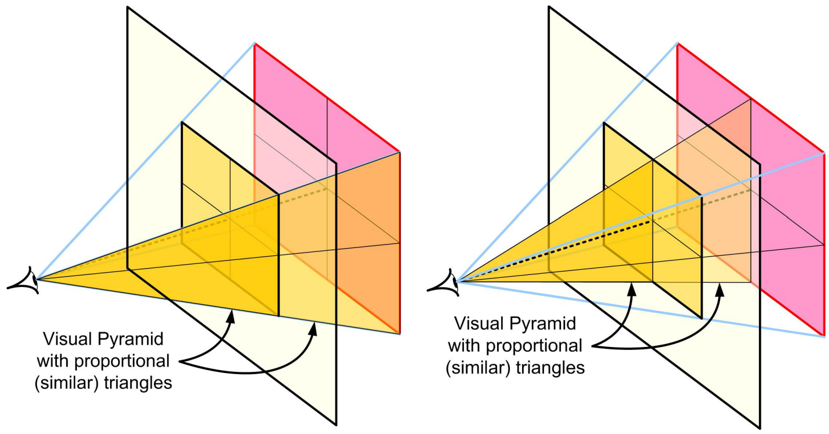


Figure 2 Proportional (Similar) Triangles on Visual Pyramid

> investigate what importance this cross-section has for us.[p.52] [JOS: see Figure 1]
>
> Planes are equidistant when the distance between one and the other is equal in all its parts. Collinear planes are those which a straight line will touch equally in ever part as in the faces of quadrangular pilasters placed in a row in a portico. These things are to be added to our treatment of the plane, intrinsic and extrinsic and centric rays and the pyramid. Let us add the axiom of the mathematicians where it is proved that if a straight line cuts two sides of a triangle, and if this line which forms a triangle is parallel to a side of the first and greater triangle, certainly this lesser triangle will be proportional to the greater.[p.52] [JOS: see Figure 2]
>
> Now let us translate our thinking to the pyramid. We should be persuaded that no quantities equidistant to the cross-section can make any alteration in the picture, because they are similar to their proportionates in every equidistant intercision. From this it follows that when the quantity with which the outline is constructed is not changed, there will be no alteration of the same outline in the picture. It is now manifest that every cross-section of the visual pyramid which is equidistant to the plane of the thing seen will be proportional to that observed plane.[pp.53-54]

Alberti's Perspective Construction

Alberti now describes his method for constructing on the painting the proportional images of the actual objects. In effect he is showing how a pavement of squares would appear under a perspective view, though I did not find his explicit statement of this until the end of his description. The following description is basically represented in Figure 3, which is assembled from diagrams in Spencer's notes to the translation and is similar to diagrams presented by others.

> Up to this point we have talked about what pertains to the power of sight and to the cross-section. Since it is not enough for the painter to know what the cross-section is, but since he should also know how to make it, we will treat of that. Here alone, leaving aside other things, I will tell what I do when I paint. First of all about where I draw. I inscribe a quadrangle of right angles, as large as I wish, which is considered to be an open window through which I see what I want to paint. Here I determine as it pleases me the size of the men in my picture. I divide the length of this man in three parts. These parts to me are proportional to that measurement called a *braccio*, for, in measuring the average man it is seen that he is about three *braccia*.[1] With these *braccia* I divide the base line of the rectangle into as many parts as it will receive. To me this base

[1] Spencer: "Book 1, Note 42. The Florentine *braccio* was slightly less than 23 inches. In the Latin text the emphasis is put on the *braccio* as a unit of measurement derived from man. The measurement is not abstract but related to man in reality and in the painting. Alberti differs from Vitruvius [*De architectura*, III, i, 2] who says that man is four cubits tall."

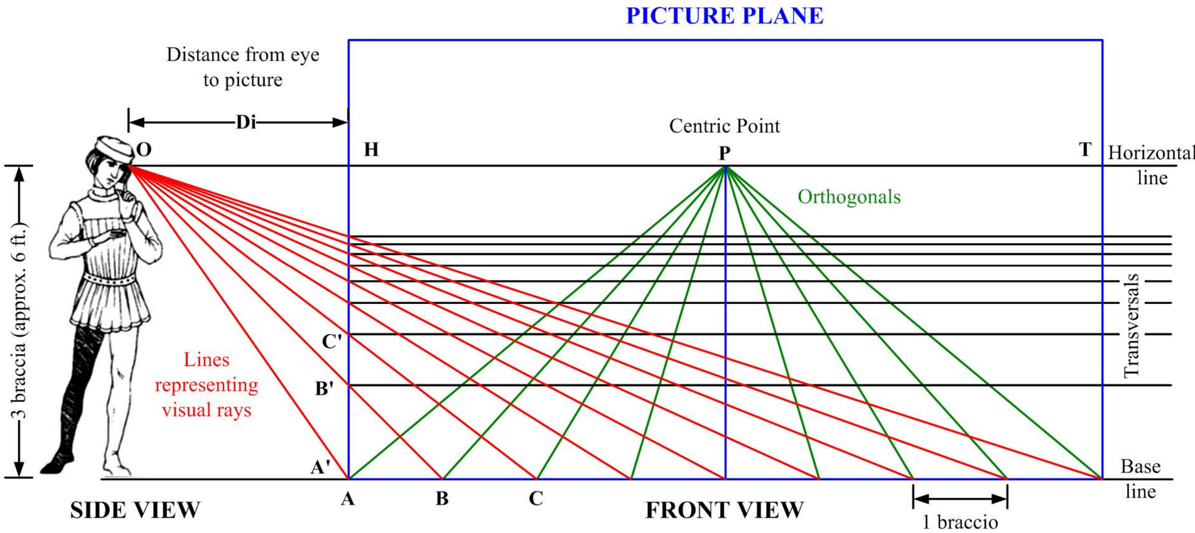
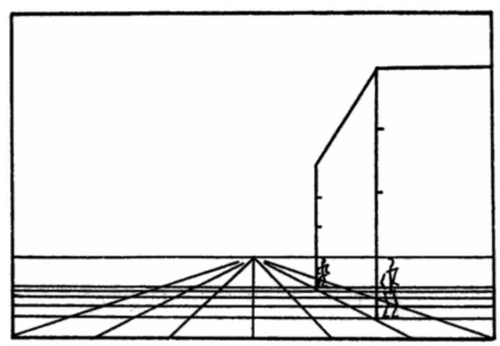


Figure 3 Alberti's Perspective Construction (from Spencer diagrams in Notes ([3] Book 1, Note 48, p.110))

line of the quadrangle is proportional to the nearest transverse and equidistant quantity seen on the pavement. Then, within this quadrangle, where it seems best to me, I make a point which occupies that place where the central ray strikes. For this it is called the centric point. This point is properly placed when it is no higher from the base line of the quadrangle than the height of the man that I have to paint there. Thus both the beholder and the painted things he sees will appear to be on the same plane.



Spencer: Book 2, Note 33, p.122

The centric point being located as I said, I draw straight lines from it to each division placed on the base line of the quadrangle. These drawn lines, [extended] as if to infinity, demonstrate to me how each transverse quantity is altered visually. [pp.55-56] …

Let us return to our subject. I find this way to be best. In all things proceed as I have said, placing the centric point, drawing the lines from it to the divisions of the base line of the quadrangle. In transverse quantities where one recedes behind the other I proceed in this fashion. I take a small space in which I draw a straight line and this I divide into parts similar to those in which I divided the base line of the quadrangle.[2] Then, placing a point at a height equal to the height of the centric point from the base line, I draw lines from this point to each division scribed on the first line. Then I establish, as I wish, the distance from the eye to the picture. Here I draw, as the mathematicians say, a perpendicular cutting whatever lines it finds. A perpendicular line is a straight line which, cutting another straight line, makes equal right angles all about it. The intersection of this perpendicular line with the others gives me the succession of the transverse quantities. In this fashion I find described all the parallels, that is, the square[d] *braccia* of the pavement in the painting. [p.57]

The confusing thing about this description, or at least the representation of it shown in Figure 3, is that the side view is superimposed on the front view. It is technically correct if the horizontal spacing of the orthogonal lines is equal to the actual spacing of the transverse lines, that is, the pavement or

[2] JOS: I don't understand this sentence, unless he means he is duplicating the divided baseline in another diagram, rather than superimposing the lines as shown in Figure 3. Alternatively, one can just omit the sentence.

floor consists of squares as Alberti finally mentions. I thought it would be easier to understand if the views were considered separately.

Suppose the transverse lines in Figure 3 are the base lines of a set of equally-spaced, square planes "equidistant" from the painted plane as shown in Figure 5. Join the top corners of the squares with green lines and the bottom corners with blue lines. Then employing Alberti's lines of sight (along the edges of his visual pyramids) and seeing where they cut the painted plane (pyramid cross-section) we obtain a nested set of squares that, if extended, would appear to converge on the centric point or what was later called the *vanishing point* (see Figure 4).

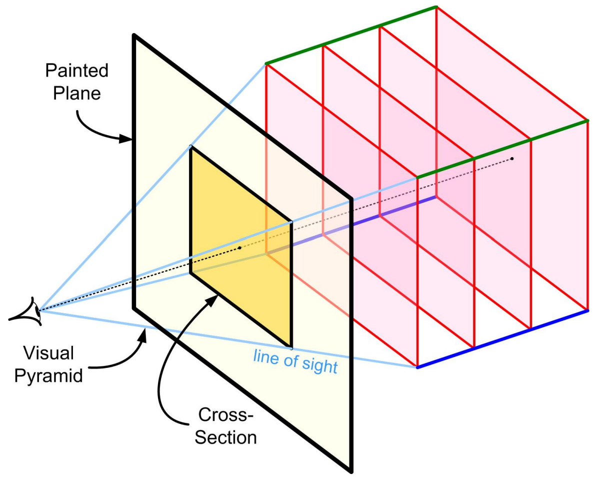


Figure 5 Joined Equidistant Square Planes

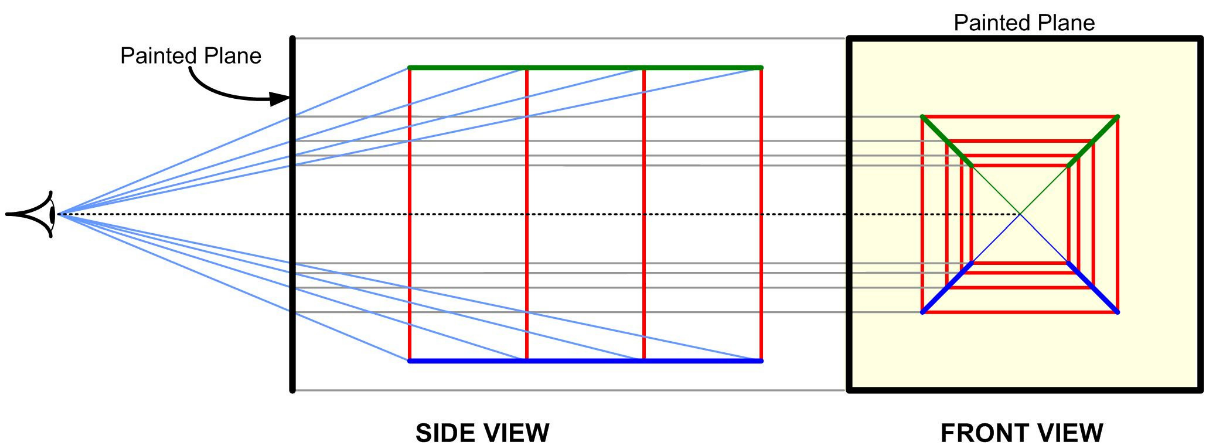


Figure 4 Perspective View of Set of Squares

Following Alberti's suggestion of using proportional (similar) triangles, we can obtain an algebraic expression in modern notation that represents the corresponding proportion (see Figure 6).

$$\frac{y}{Y} = \frac{d}{d+D} \quad \text{or} \quad y = \frac{d}{d+D}Y$$

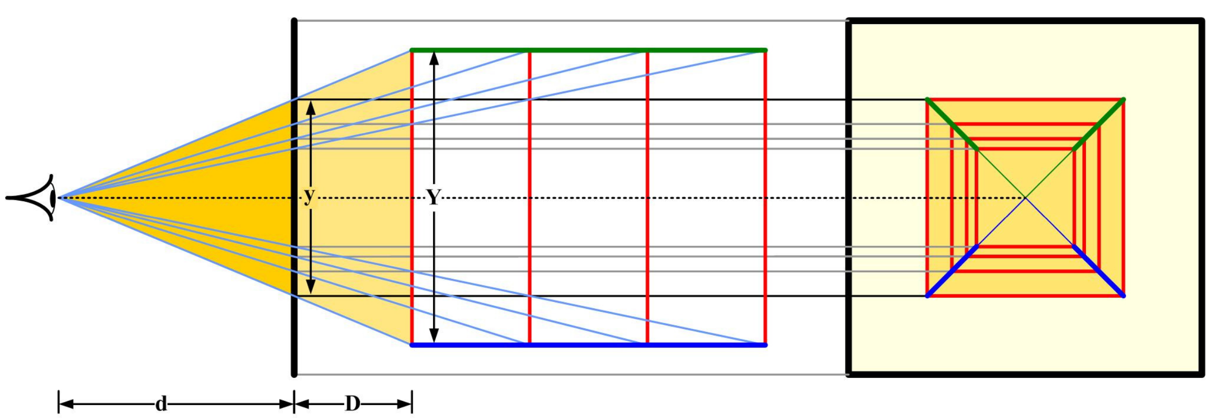


Figure 6 Computations for Perspective Size Reductions Based on Similar Triangles.

As the actual objects recede from the painted plane (as D grows larger), their image heights y shrink—eventually to zero, at the vanishing point.

Now the interesting question is what happens when the objects are not all on equidistant planes. I find Alberti's discussion obscure at this point. So I just took the case of a cube, joining two of the squares, and rotated it 45° relative to the painted plane (see Figure 7 and Figure 8).

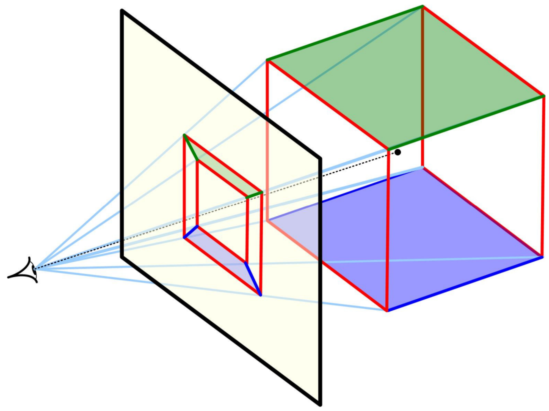


Figure 7 Cube with Equidistant (Red) Planes

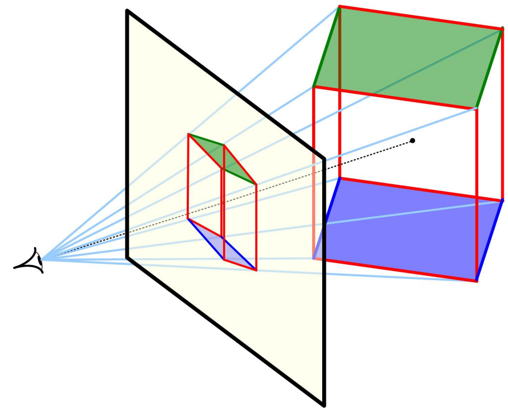


Figure 8 Cube Rotated 45°

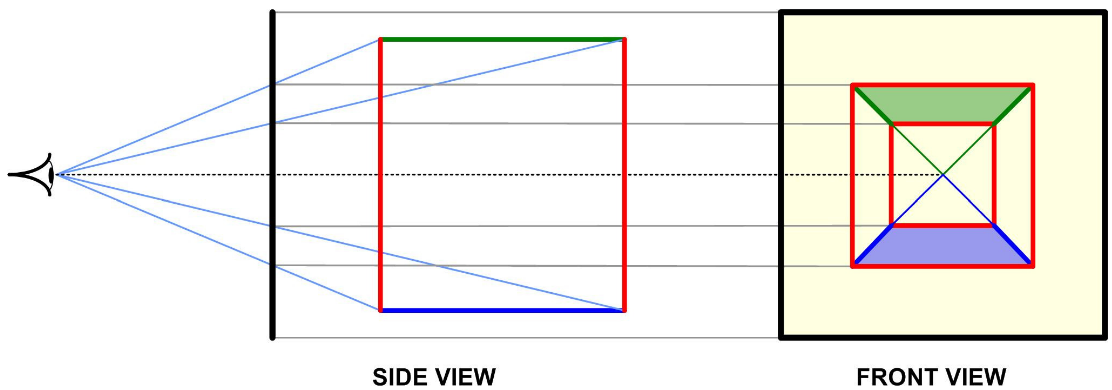


Figure 9 Cube with Equidistant Planes in Perspective

Figure 9 shows the "equidistant planes" cube under the perspective lines of sight we have been using. Figure 10 shows the rotated cube using the same type of argument. That is, the vertical edge of the cube closest to the painted plane can be thought of as being in one virtual equidistant plane, the two outer vertical edges of the rotated cube can be thought of as being in a second equidistant plane (since we rotated 45°) and the vertical edge furthest from the painted plane can be thought of as being

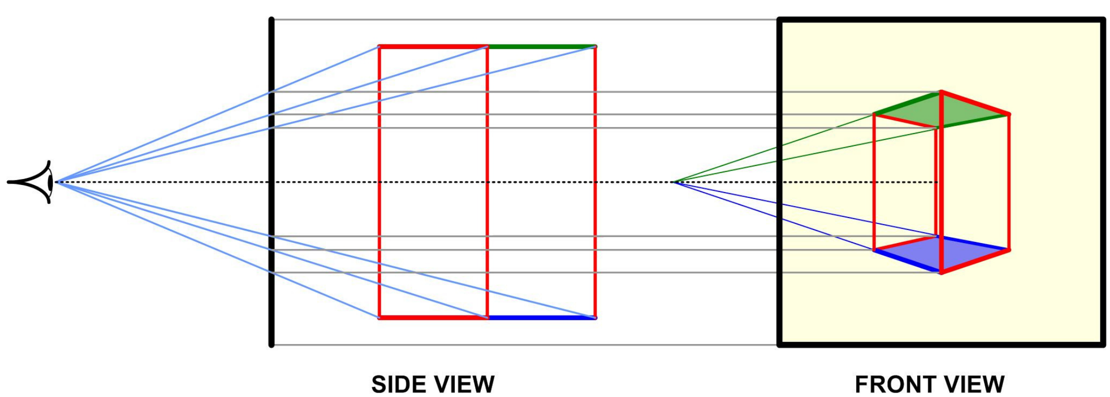


Figure 10 Perspective View of Rotated Cube

in a third equidistant plane. Drawing sight lines as before to the top and bottoms of these edges and seeing where they cut the painted plane leads us to the Front View in Figure 10 of the rotated cube.

An interesting feature shows up in the perspective front view of the rotated cube, namely, a new vanishing point exists for the extensions of the top and bottom edges of the cube. I don't know of any obvious mathematical way of proving this via contemporary plane geometry (I used trigonometry before), but the new vanishing point would have been evident to any Renaissance painter who was carefully following Alberti's instructions. So it looks like a fair number of properties of the perspective map can be inferred from the methods of construction described by Alberti.

References

[1] Wootton, David, *The Invention of Science: A New History of the Scientific Revolution*, Harper, 784pp, 2015

I can't recommend this bloated and verbose book (almost 800 pages!). Even though I only delved into a small part in the chapter on "The Mathematization of the World", the error I found there tainted the whole enterprise. On page 177 Wootton states,

> Euclid lacked the number zero, which was introduced into Western Europe in the early thirteenth century with what we call the Arabic numerals (actually, it is the only one of the ten numerals that is Arabic; the others are Indian).

Wootton provides no reference for this. Actually, *all* the numerals (and numbers) came from the Indians for a decimal (base ten) system that included zero as a *number* and not just a place-holder:

> In 628 CE, [Indian] astronomer-mathematician Brahmagupta wrote his text *Brahma Sphuta Siddhanta* which contained the first mathematical treatment of zero. He defined zero as the result of subtracting a number from itself, postulated negative numbers and discussed their properties under arithmetical operations. (*Wikipedia*)

There are numerous other references that could be cited besides *Wikipedia*, such as Kline's *Mathematical Thought from Ancient to Modern Times* (1972). Wootton's reference to the thirteenth century probably refers to Leonardo of Pisa (Fibonacci) and his book *Liber Abacci* (1202), which introduced to the West almost 600 years later the Hindu-Arabic number system and Hindu-Arabic methods of arithmetic, including the manipulation of fractions (actual ratios of integers with numerator and denominator). Clearly the bloat of the book meant that Wootton could not master all the details. So I have no confidence that anything else he says is true. I ended up using the book as a stimulus to investigate a subject further.

I just noticed another misleading item on page 208. Wootton is discussing world maps and mentions "Mercator's projection (1599)" with surprisingly no further amplification, given his usual verbosity. As I make plain in my "Mercator Projection Balloon" post, Mercator's *map* using the projection was in 1569, but the mathematics of the *projection* was explained by *Wright* in his 1599 book. So technically Wootton may claim his date was correct, but it is highly misleading in the context of his discussion of maps.

[2] King, Ross, *Brunelleschi's Dome: How a Renaissance Genius Reinvented Architecture*, Walker & Co, 199pp, 2000

This is a superb book—well-written with fascinating and surprising information.

> *Brunelleschi's Dome* [of the Cathedral of Santa Maria del Fiore in Florence, Italy] is the story of how a Renaissance man bent men, materials, and the very forces of nature to build an architectural wonder. Not a master mason or carpenter, Filippo Brunelleschi was a goldsmith and clock maker. Over twenty-eight years, he would dedicate himself to solving puzzles of

the dome's construction. In the process, he did nothing less than reinvent the field of architecture. He engineered the perfect placement of brick and stone ([inventing] some among the most renowned machines of the Renaissance) to carry an estimated seventy million pounds hundreds of feet into the air, and designed the workers' platforms and routines so carefully that only one man died during the decades of construction. This drama was played out amid plagues, wars, political feuds, and the intellectual ferments of Renaissance Florence — events Ross King weaves into a story to great effect." (Book Blurb)

[3] Alberti, Leon Battista. *On Painting*. [First appeared 1435-36] Translated with Introduction and Notes by John R. Spencer, New Haven, Yale University Press, 1970 [First printed 1956]. (http://www.noteaccess.com/Texts/Alberti/, retrieved 6/7/2019)