Star Polygon Problem

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This is problem #25 from the UKMT 2014 Senior Challenge.

Figure 1 shows a tile in the form of a trapezium [trapezoid], where $\alpha = 83\frac{1}{3}°$. Several copies of the tile placed together form a symmetrical pattern, part of which is shown in Figure 2. The outer border of the complete pattern is a regular ‘star polygon’. Figure 3 shows an example of a regular ‘star polygon’.

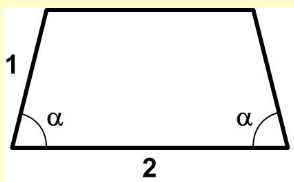


Figure 1

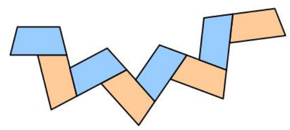


Figure 2

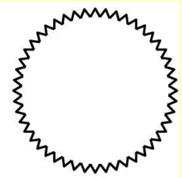


Figure 3

How many tiles are there in the complete pattern?

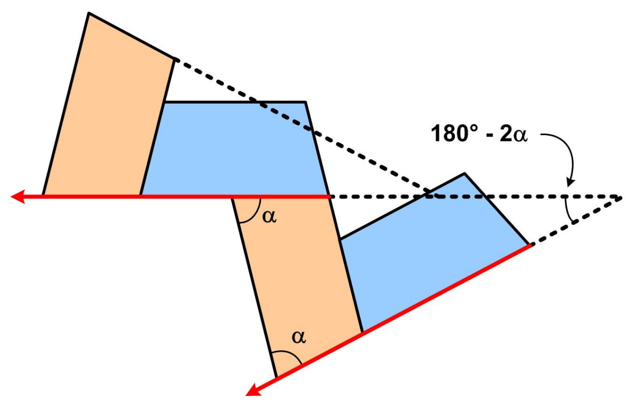
A 48 B 54 C 60 D 66 E 72

My Solution

The key to the solution is to notice that each pair of adjacent tiles repeats in the border pattern after a rotation of $180° - 2\alpha = 180° - 2 \cdot 83\frac{1}{3}° = 13\frac{1}{3}° = 40°/3$. Since there must be an integral number of these wedges around the circle (alternatively, the red arrow in the figure must rotate 360° as it traverses the border and returns to its original position), we divide 360° by $13\frac{1}{3}°$:

$$360/(40/3) = 27.$$

So there are 27 wedges in the border and each wedge has two tiles. So the number of tiles in the border is 54 (Answer B).

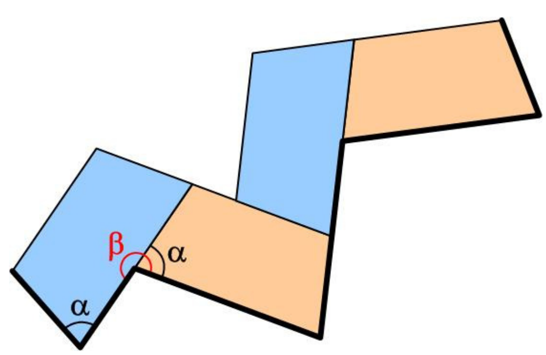


Tile Pair in Border Wedge

UKMT Solution 1

The figure on the right shows part of the complete pattern. The heavier lines form part of the border of the ‘star polygon’. The complete pattern is made up of an even number of tiles. Let the number of tiles be 2n, where n is a positive integer.

The ‘star polygon’ has 2n edges and so it has 2n internal angles. From the figure we see that n of these angles have size α and n of them have size β, where $\beta = \alpha + 180°$. Therefore the sum of the internal angles of the ‘star polygon’ is $n\alpha + n(\alpha + 180°)$, that is,



UKMT Solution 1

$n(2\alpha + 180°)$. Since $\alpha = 83\frac{1}{3}°$, $2\alpha + 180° = 1040°/3$. So the sum of the internal angles is $n \times 1040°/3$. On the other hand, the sum of the internal angles of a polygon with $2n$ edges is $(2n - 2)\times 180°$.

The two expressions we have obtained for the sum of the internal angles are equal, and so we have

$$(2n - 2) \times 180 = n \times 1040/3.$$

This last equation may be rearranged to give

$$(360 - 1040/3) \times n = 360$$

that is, $$n \times 40/3 = 360$$

from which it follows that $$n = 360 \times 3/40 = 27$$

We deduce that the 'star polygon' has $2n$ = 54 sides.

UKMT Solution 2

For Investigation. We consider an alternative method for answering Question 25. Consider moving anticlockwise around the perimeter of the 'star polygon'. At each vertex you either turn anticlockwise through the angle $(180° - \alpha)$ or clockwise through the angle α.

(a) Suppose that the 'star polygon' has $2n$ edges. What is the total anticlockwise angle that you turn through as you move anticlockwise around it?

(b) Since in going completely round the 'star polygon' anticlockwise you turn through an angle 360°, what value does this give for n?

Remark. This problem was inspired by some floor tiling in the Church on Spilled Blood in St Petersburg. This church, also called the Church of the Saviour on Spilled Blood or the Cathedral of the Resurrection of Christ, is in St Petersburg in Russia.