Fibonacci Fandango

1 June 2019

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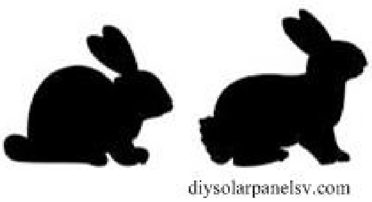

This is from the UKMT Senior Challenge of 1999.

What is the sum to infinity of the convergent series

$$\frac{1}{2}+\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\frac{5}{32}+\frac{8}{64}+\frac{13}{128}+\frac{21}{256}+\frac{34}{512}+\ldots ?$$

A 7/4 B 2 C $\sqrt{5}$ D 9/4 E 7/3

Solution

First, write the series and its sum *S* as

$$\frac{1}{2}+\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\frac{5}{32}+\frac{8}{64}+\frac{13}{128}+\frac{21}{256}+\frac{34}{512}+\ldots=\sum_{n=1}^{\infty}\frac{a_n}{2^n}=\sum_{n=1}^{\infty}b_n=S \tag{1}$$

where the numerators a_n satisfy the Fibonacci sequence relationship

$$a_n = a_{n-1} + a_{n-2}$$

for n = 3, 4, …. I got side-tracked at first looking at ratios of successive b_n derived from the a_n, which would lead to a relationship involving the golden mean $(1\pm\sqrt{5})/2$. But that only helps with establishing convergence and not the sum. The Fibonacci relation does induce a direct relationship on the b_n, however, which turns out to be useful:

$$b_n=\frac{a_n}{2^n}=\frac{a_{n-1}+a_{n-2}}{2^n}=\frac{1}{2}\frac{a_{n-1}}{2^{n-1}}+\frac{1}{4}\frac{a_{n-2}}{2^{n-2}}=\frac{1}{2}b_{n-1}+\frac{1}{4}b_{n-2}$$

for n = 3, 4, …. Then

$$\sum_{n=3}^{\infty}b_n=\frac{1}{2}\sum_{n=2}^{\infty}b_n+\frac{1}{4}\sum_{n=1}^{\infty}b_n$$

or, from equation (1)

$$\left(S-\frac{1}{2}-\frac{1}{4}\right)=\frac{1}{2}\left(S-\frac{1}{2}\right)+\frac{1}{4}S$$

and so

$$S = 2 \quad \text{(Answer B)}$$

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