Cube Roots Problem

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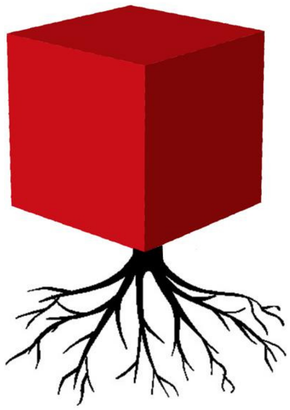

In a June *Chalkdust* book review[1] of Daniel Griller's second book, *Problem solving in GCSE mathematics*, Matthew Scroggs presented the following problem #65 from the book (without a solution):

Solve

$$\sqrt[3]{\frac{4x}{15}} + 2 = \sqrt[3]{\frac{9x}{10}}$$

Scroggs's initial reaction to the problem was "it took me a while to realise that I even knew how to solve it."

Mind you, according to *Wikipedia*, "GCSEs [General Certificate of Secondary Education] were introduced in 1988 [in the UK] to establish a national qualification for those who decided to leave school at 16, without pursuing further academic study towards qualifications such as A-Levels or university degrees." My personal feeling is that any student who could solve this problem should be encouraged to continue their education with a possible major in a STEM field.

Solution

My initial reaction was similar to Scroggs's. I knew that cubing both sides would lead to an unbelievably complicated rat's nest of cubic fractional powers that probably could not be unraveled. So I thought I would try the obnoxious and inelegant trial-and-error approach of guessing a solution for x. If on the left hand side I multiplied the 4 by 2 and cancelled the 15, namely $x = 2\cdot 15$, I would get a perfect cube of 8 and so cube root of 2. Similarly, on the right hand side if I multiplied the 9 by 3 and cancelled the 10, namely $x = 3\cdot 10$, I would get a perfect cube of 27 and so cube root of 3. Amazingly, this approach gave the same choice for x, namely 30. Alas, plugging this value for x into the equation gave $2 + 2 = 3$. So failure.

But I kept fiddling with the idea of choosing an x so that perfect cubes would arise and I stumbled on the following successful approach.

$$\sqrt[3]{\frac{4x}{15}} = \sqrt[3]{\frac{4\cdot 2\cdot x}{2\cdot 15}} = 2\sqrt[3]{\frac{x}{30}} \quad \text{and} \quad \sqrt[3]{\frac{9x}{10}} = \sqrt[3]{\frac{9\cdot 3\cdot x}{3\cdot 10}} = 3\sqrt[3]{\frac{x}{30}}$$

So let $y = \sqrt[3]{x/30}$. Then $2y + 2 = 3y$, so $y = 2$ and $y^3 = 8$ or $x = 8\cdot 30 = 240$. Amazing!

This is a wonderfully ingenious problem that speaks highly for the imagination of the author Daniel Griller. And Matthew Scroggs in his review says, "The book contains many puzzles like this that will get even the highest achieving GCSE students to think about what they know *in a different way* [my emphasis]." Alas, things may have changed over the years since I was teaching, but any problem I found refreshing and "different" was usually death to the student. They were just barely understanding the basics, and anything out of the ordinary was too much of a challenge—especially on a timed test. In fact, the pleasure that a confident mathematician such as Scroggs feels from the initial sensation of an impossible problem is a terrifying mental block to a novice under pressure of an

[1] http://chalkdustmagazine.com/blog/review-of-problem-solving-in-gcse-mathematics/

exam. Solving a challenging problem at leisure is a far different experience from trying to solve one under pressure. Like I said before, if a student handles problems of this type on an exam, then they are definitely deserving of all the encouragement and financial help they can get to continue their education. They would have shown they have real talent.