Triangle Acute-Angle Problem

(8 October 2018)

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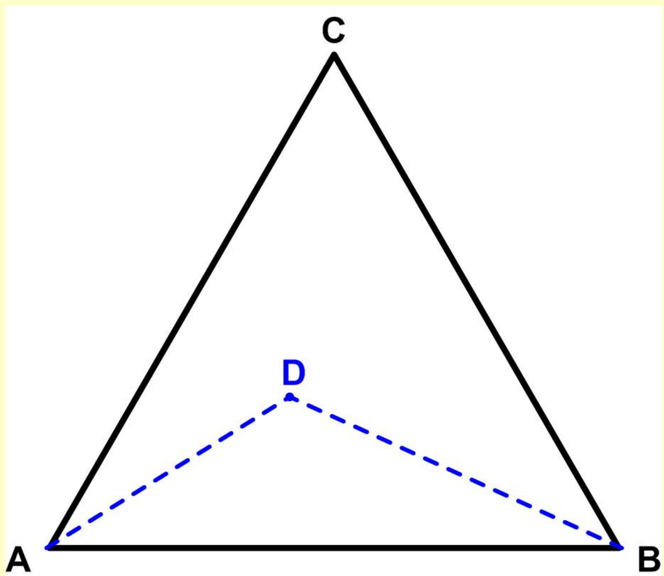
Catriona Shearer Retweeted the following problem from Antonio Rinaldi @rinaldi6109 (https://twitter.com/rinaldi6109):

My little contribution to @Cshearer41 October 7, 2018

(https://twitter.com/rinaldi6109/status/1048892029248458752)



A point D is randomly chosen inside the equilateral triangle ABC. Determine the probability that the triangle ABD is acute-angled.

Solution

The first thing to notice is that so long as vertex D is inside triangle ABC, the angles at vertices A and B in triangle ABD will always be acute. Therefore, we only need to consider where the angle at D will be acute, that is, less than a right angle (90° or π/2 radians). The boundary of such a region will be where D is exactly a right angle, which would be a semicircle with diameter AB (Figure 1).

If we assume (without loss of generality, wlog) that the equilateral triangle has sides of length 4 (and altitude $2\sqrt{3}$), then the equations for the semicircle and line CB are as shown in Figure 1. The intersection of these curves is the point $(1, \sqrt{3})$, the vertex of a small equilateral triangle with dimensions half those of the original (and therefore with area ¼ the original of $4\sqrt{3}$). The area where angle D is acute is shown in green. We now compute its value.

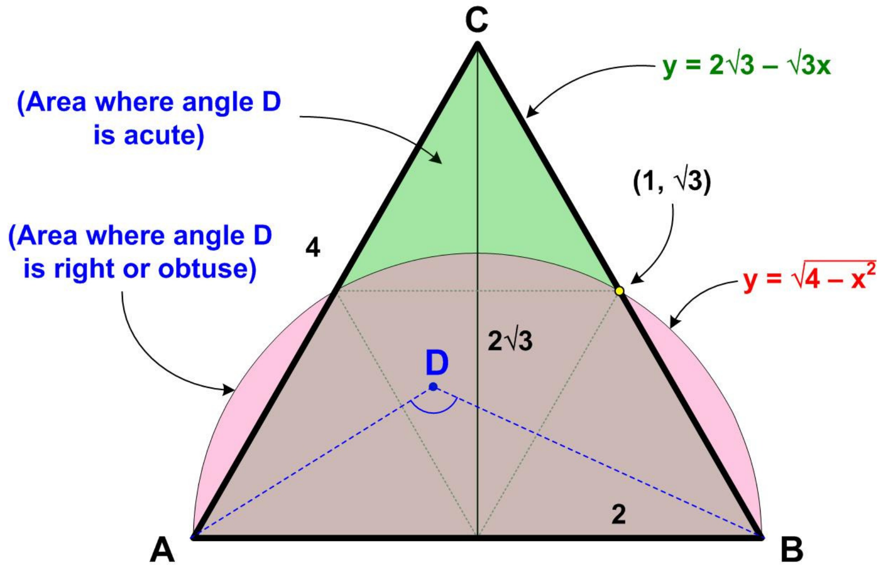


Figure 1 Semi-circle area of exclusion

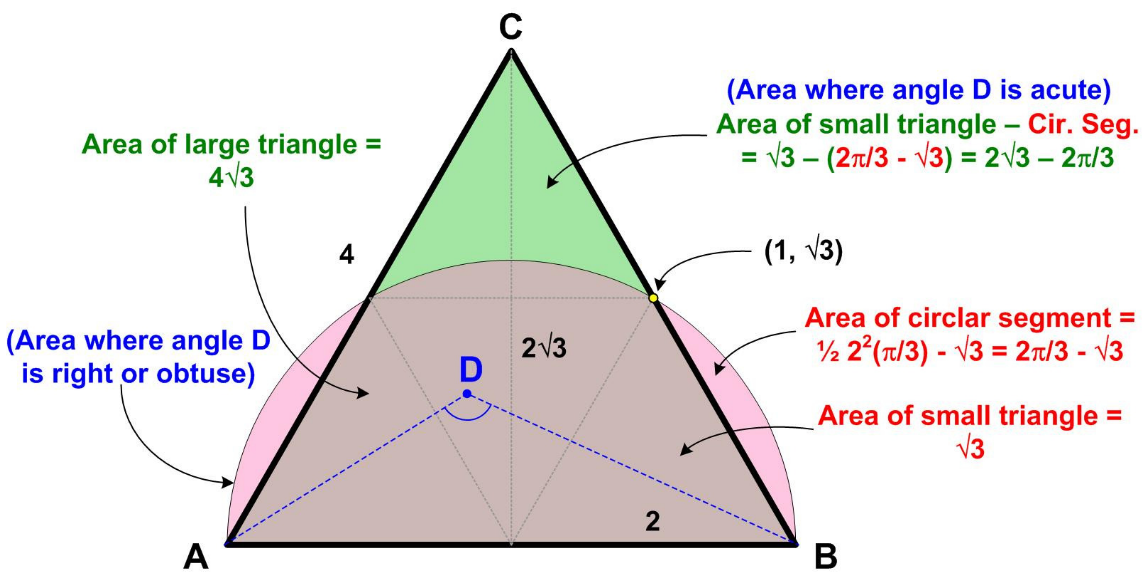


Figure 2 Area Computations

Figure 2 shows the computations for the various areas with the acute-angle region given by

$$2\sqrt{3} - 2\pi/3,$$

so that the probability of the point D landing in this region of the large equilateral triangle is

$$(2\sqrt{3} - 2\pi/3) / 4\sqrt{3} = \frac{1}{2} - \pi / 6\sqrt{3} \approx .198 \approx 20\%$$