Rising Sun

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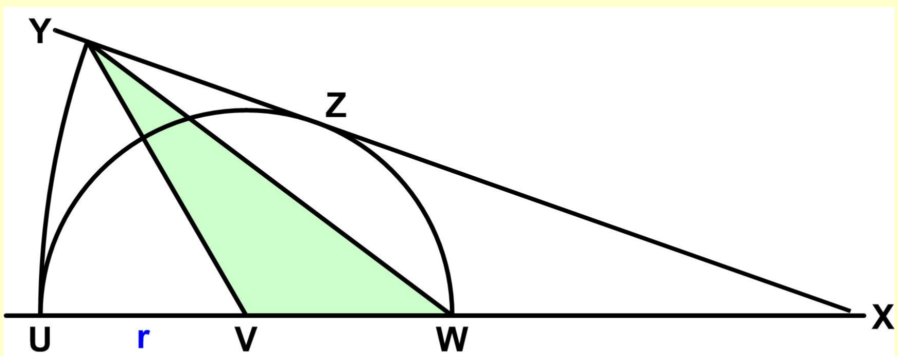
Here is a problem from the UKMT Senior (17-18 year-old) Mathematics Challenge for 2012:

A semicircle of radius *r* is drawn with centre *V* and diameter *UW*. The line *UW* is then extended to the point *X*, such that *UW* and *WX* are of equal length. An arc of the circle with centre *X* and radius 4r is then drawn so that the line *XY* is tangent to the semicircle at *Z*, as shown. What, in terms of *r*, is the area of triangle *YVW*?



Solution

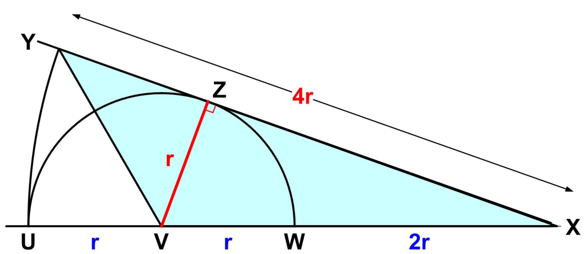


Figure 1 Area of Large Blue Triangle

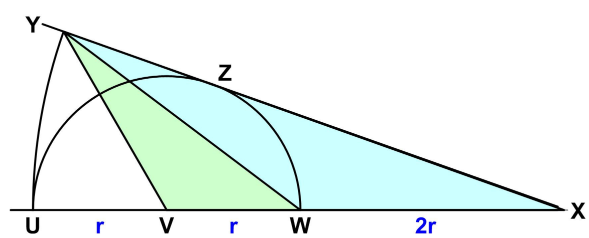


Figure 2 Area of Small Green Triangle

As shown in Figure 1, draw a radius of the semicircle from the vertex V of the blue triangle to the base YX. Because the base YX is tangent to the semicircle, the radius is perpendicular to YX and so is an altitude of the triangle. Moreover, YX = UX = 4r, so the area of the blue triangle is

$$\tfrac{1}{2}\, r\, (4r) = 2\, r^2$$

Now consider Figure 2, which shows the desired green triangle. It has the same altitude as the blue triangle (perpendicular from Y down to UX) and its base r is 1/3 the base VX = 3r of the blue triangle, so its area is 1/3 the area of the blue triangle, or

$$\text{Area of green triangle} = {}^{2}/_{3}\, r^2$$

Alternative UKMT Solution

One of the UKMT solutions was the same as I got above. But they also showed another that I thought was a bit more complicated. It used trigonometry instead of just geometry.

Let the perpendicular from Y meet UV at T and let $\angle ZXV = \alpha$. Note that $\angle VZX = 90°$ as a tangent to a circle is perpendicular to the radius at the point of contact. Therefore $\sin\alpha = \frac{r}{3r} = \frac{1}{3}$. Consider triangle YTX: $\sin\alpha = \frac{YT}{YX}$. So $YT = YX\sin\alpha = \frac{4r}{3}$. So the area of triangle $YVW = \frac{1}{2} \times VW \times YT = \frac{1}{2} \times r \times \frac{4r}{3} = \frac{2r^2}{3}$.