Radical Radicals

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This is another UKMT Senior Challenge problem, but for the year 2005. I thought it was diabolical and hadn't a clue how to solve it. Even after reading the solution, I don't think I could have come up with it. I take my hat off to anyone who solves it.

Which of the following is equal to

$$\frac{1}{\sqrt{2005+\sqrt{2005^2-1}}}\,?$$

A $\sqrt{1003}-\sqrt{1002}$ B $\sqrt{1005}-\sqrt{1004}$ C $\sqrt{1007}-\sqrt{1005}$

D $\sqrt{2005}-\sqrt{2003}$ E $\sqrt{2007}-\sqrt{2005}$

UKMT Solution

I will just provide the UKMT solution and then whine about it afterward.

$$\begin{aligned}
\frac{1}{\sqrt{2005+\sqrt{2005^2-1}}} &= \frac{1}{\sqrt{1003+1002+\sqrt{(2005+1)(2005-1)}}} \\
&= \frac{1}{\sqrt{\sqrt{(1003)^2}+2\sqrt{1003}\sqrt{1002}+\sqrt{(1002)^2}}} \\
&= \frac{1}{\sqrt{\left(\sqrt{1003}+\sqrt{1002}\right)^2}} \\
&= \frac{1}{\sqrt{1003}+\sqrt{1002}} \\
&= \frac{\left(\sqrt{1003}-\sqrt{1002}\right)}{\left(\sqrt{1003}+\sqrt{1002}\right)\left(\sqrt{1003}-\sqrt{1002}\right)} \\
&= \frac{\left(\sqrt{1003}-\sqrt{1002}\right)}{1003-1002} \\
&= \sqrt{1003}-\sqrt{1002}
\end{aligned}$$

So the answer is A $\sqrt{1003}-\sqrt{1002}$.

I find this astonishing. I guess I will add fussing with radicals to the list of math subjects I find frustrating, such as number theory with its simple statements and impossible proofs, probability and statistics with their incomprehensible and irreproducible explanations, and combinatorics with their echoes of tedious elementary school arithmetic. This is my own failing, since these latter subjects are matters of profound and current mathematics.

Actually there is historical precedent for the unusual and obscurantist behavior of radicals. It began in the beginning with the attempt to solve polynomial equations.

From *Wikipedia* ([1]) “Nested radicals appear in the algebraic solution of the cubic equation. Any cubic equation can be written in simplified form without a quadratic term, as $x^3 + px + q = 0$, whose general solution for one of the roots is [from Cardano (1501–1576)]:

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \text{ ,”}$$

And from ([2], p.20, Exercise 6) “Consider the equation $x^3 + x - 2 = 0$. Note that $x = 1$ is a root. Use Cardan’s formulas (carefully) to derive the surprising formula

$$1 = \sqrt[3]{1 + \frac{2}{3}\sqrt{\frac{7}{3}}} + \sqrt[3]{1 - \frac{2}{3}\sqrt{\frac{7}{3}}} \text{ ,”}$$

(There are a bunch of issues about complex numbers that I am ignoring.)

This crazy result in using the closed form formula to solve such polynomial equations created consternation in the mathematicians pioneering the exploration of the subject. Introducing complex numbers and trigonometric functions to get complete solutions only seems to add to the confusion, at first. But eventually they provided a simpler way to organize and understand the issues and solutions.

So the above problem is just a taste of the kind of mess that can happen with radicals.

References

[1] “Nested Radical,” *Wikipedia*, (https://en.wikipedia.org/wiki/Nested_radical, retrieved 5/25/2019)

[2] Cox, David, *Galois Theory*, 2nd ed., John Wiley, New Jersey, 2012