# Parallelogram Cosine Problem

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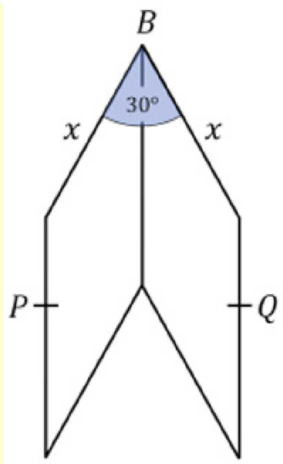
Another challenging problem from Presh Talwalkar. I certainly could not have solved it on a timed test at the age of 16.




## One Of The Hardest GCSE Test Questions – How To Solve The Cosine Problem

Posted May 8, 2019 By Presh Talwalkar.

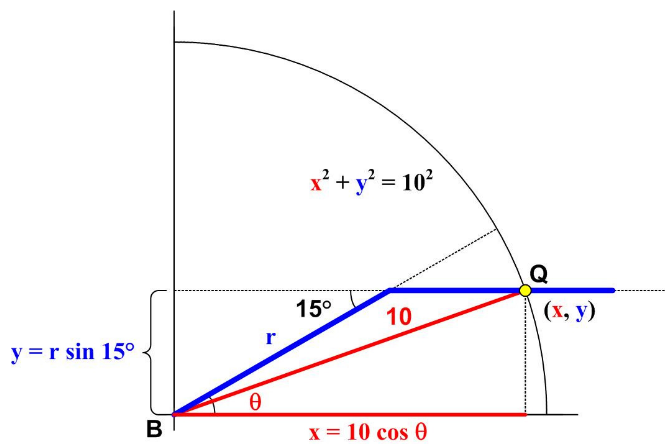
Construct a hexagon from two congruent parallelograms as shown. Given BP = BQ = 10, solve for the cosine of PBQ in terms of x.

This comes from the 2017 GCSE[1] exam, and it confused many people. I received many requests to solve this problem, and I thank Tom, Ben, and James for suggesting it to me.

It turns out that Talwalkar has a slick solution, but I rather suspect I would not have thought of it. Instead, I had a different approach that I will motivate as I go along.

## My Solution

First, I realized there was an arbitrariness about where the points P and Q fell, but they were at least 10 units from B. That suggested a circle of radius 10 centered at B. That meant that P and Q were the intersections of the circle and straight lines along the sides of the parallelogram that cut the slant sides of the parallelogram x units from B.




I rotated the figure 90° counterclockwise and only looked at one of the parallelograms, as shown in the figure to the right. Given the use of a circle, I relabeled things using more traditional variables, exchanging x for the radial distance r along the 15° line. Then the variable y represents the distance between the other two edges of the parallelogram determined by the length x (which is constrained so that there are actually intersection points P and Q). From the diagram I can compute cos θ, where θ is half the ultimate angle desired. Namely, cos θ = x/10, where from the circle and double-angle trig identities

$$x^2 = 10^2 - y^2 = 10^2 - r^2 \sin^2 15° = 10^2 - r^2 (1 - \cos 30°)/2 = 10^2 - r^2 (1 - \surd 3/2)/2$$

[1] JOS: (*Wikipedia*) In the United Kingdom, the General Certificate of Secondary Education (GCSE) is an academic qualification, …. Each GCSE qualification is in a particular subject, and stands alone, but a suite of such qualifications (or their equivalent) is generally accepted as the record of achievement at the age of 16…. Studies for GCSE examinations generally take place over a period of two or three academic years (depending upon the subject, school, and exam board), starting in Year 9 or Year 10 for the majority of students, with examinations being sat at the end of Year 11.

or $x^2/10^2 = 1 - r^2 (1 - \sqrt{3}/2)/200$

Then $\cos 2\theta = 2 \cos^2 \theta - 1 = 2 (x/10)^2 - 1 = 1 - 2r^2 (1 - \sqrt{3}/2)/200$

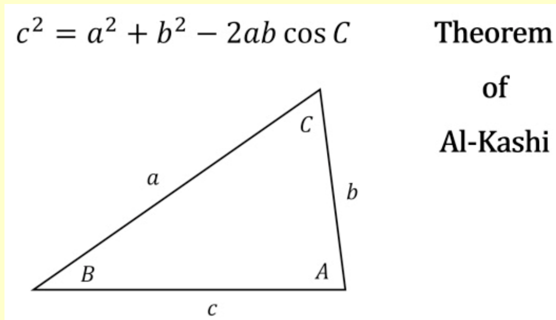
or, swapping r back to x,

$$\cos PBQ = 1 - x^2(2 - \sqrt{3})/200$$

# Talwalkar's Solution

## Answer To One Of The Hardest GCSE Test Questions

Recall in any triangle $c^2 = a^2 + b^2 - 2ab \cos C$



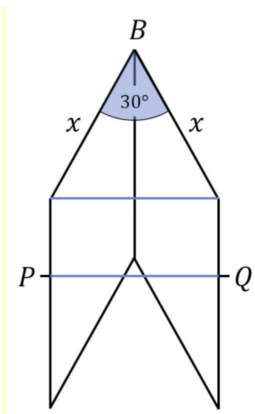

This is often called the "law of cosines." But in my research, I found a fun fact on Wikipedia[2]:

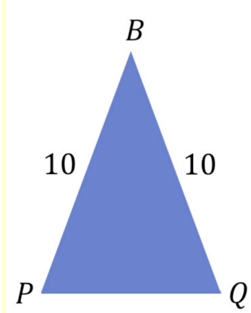
*In France, the law of cosines is named Théorème d'Al-Kashi (Theorem of Al-Kashi), as al-Kashi was the first to provide an explicit statement of the law of cosines in a form suitable for triangulation.*

I think it makes sense to call the identity Al-Kashi's Theorem, and I will try to call it that henceforth.

### The Solution

The problem is a test of applying Al-Kashi's Theorem two times. First draw $PQ$ and notice it has the same length if we shift the line segment to connect the corners of two parallelograms, as shown here:




We can use Al-Kashi's Theorem in the upper triangle, noting the side opposite angle $B$ has the same length as $PQ$. This gives:

$$PQ^2 = x^2 + x^2 - 2(x)(x) \cos 30°$$

$$PQ^2 = 2x^2 - 2x^2(0.5\sqrt{3})$$

$$PQ^2 = x^2(2 - \sqrt{3})$$

Now let's focus on triangle $PBQ$, recalling that we are given $BP = BQ = 10$. We can use Al-Kashi's Theorem to get:

$$PQ^2 = 10^2 + 10^2 - 2(10)(10)\cos PBQ$$

$$PQ^2 = 200 - 200 \cos PBQ$$

We now have two expressions equal to $PQ^2$, so we can set them equal to each other. We get:

$$x^2(2 - \sqrt{3}) = 200 - 200 \cos PBQ$$

[2] https://en.wikipedia.org/wiki/Jamsh%C4%ABd_al-K%C4%81sh%C4%AB

Subtract 200 from both sides and then divide by -200 and we get the answer:

$$\cos PBQ = 1 - x^2(2 - \sqrt{3})/200$$

It's a tricky problem since you have to use the law of cosines twice, and you have to keep track of a lot of algebra. I don't think I'd solve this in a timed test. But it is a fun problem nevertheless!