The Weight Problem of Bachet de Méziriac

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The following is a famous problem of Bachet as recounted by Dörrie ([1] p.7):

The Weight Problem of Bachet de Méziriac

A merchant had a forty-pound measuring weight that broke into four pieces as the result of a fall. When the pieces were subsequently weighed, it was found that the weight of each piece was a whole number of pounds and that the four pieces could be used [in a balance scale] to weigh every integral weight between 1 and 40 pounds [when we are allowed to put a weight in either of the two pans]. What were the weights of the pieces?

(This problem stems from the French mathematician Claude Gaspard Bachet de Méziriac (1581-1638), who solved it in his famous book *Problèmes plaisants et délectables qui se font par les nombres*, published in 1624.)

The authorship of this problem is also noted by H. E. Dudeney ([2] p.109) and W. W. Rouse Ball ([3] p.34)[1]

Solution

The basic idea of using a balance scale is that we put the object to be weighed in the right-hand pan and one or more of the known weights in the left-hand pan until the two sides balance, that is, have the same weight.

We would like a mechanism that would use a weight or not, which is usually represented by a 1 or a 0. That suggests a binary number or number base 2. For example, an object weighing 23 pounds could be represented via a binary number as

$$23_{10} = 16 + 4 + 2 + 1 = 1\cdot 2^4 + 0\cdot 2^3 + 1\cdot 2^2 + 1\cdot 2^1 + 1\cdot 2^0 = 10111_2$$

We could measure its weight with the four weights 16, 4, 2, and 1 pounds. If we added an 8 pound weight, we could measure the weights of any object up to 31 pounds. To reach 40, we would need to add a 32 pound weight. That would be 6 weights in all. But our problem only has 4 weights.

We need to consider heavier weights so that we can use fewer of them. So look at base 3 or ternary (trinary) numbers.

$$23_{10} = 18 + 3 + 2 = 2\cdot 3^2 + 1\cdot 3^1 + 2\cdot 3^0 = 212_3$$

[1] I just discovered that Leonardo of Pisa (aka Fibonacci) had this problem in his 1202 AD book *Liber Abaci* p.297 ([4] p.420 in Sigler translation). He called it "On IIII Weights Weighing Forty Pounds." I was thrown off at first by the IIII, until I realized Leonardo was using Roman numerals and not in the IV form. So instead of being 400 years old, the problem is at least 800 years old! In fact, I am finding a number of problems came from Leonardo's book. And of course, most of those probably came from the Arabs and Persians from even earlier centuries. Yet another example of the durability of mathematics.

To reach 40 we would need to add a 27 pound weight. Then we would have 1, 3, 9, and 27 pound weights as our set. But unfortunately, we apparently need 2 of each (or maybe only 1 of the 27 pound weights to reach 40). So that would be 7 weights, a degradation of 1 over the binary system of 6 weights.

However, $2 = 3 - 1 = 10_3 - 1_3$. This means we could write 23 as

$$
\begin{aligned}
23_{10} &= 2 \cdot 3^2 + 1 \cdot 3^1 + 2 \cdot 3^0 \\
&= (3-1) \cdot 3^2 + 1 \cdot 3^1 + (3-1) \cdot 3^0 \\
&= 1 \cdot 3^3 - 1 \cdot 3^2 + 2 \cdot 3^1 - 1 \cdot 3^0 \\
&= 1 \cdot 3^3 - 1 \cdot 3^2 + (3-1) \cdot 3^1 - 1 \cdot 3^0 \\
&= 1 \cdot 3^3 + (1-1) \cdot 3^2 + - 1 \cdot 3^1 - 1 \cdot 3^0 \\
&= 1000_3 - 11_3 \ (= 27_{10} - 4_{10})
\end{aligned}
$$

This expression is equivalent of putting a 27 pound weight in the left-hand pan and a 3 pound and 1 pound weight in the right-hand pan along with the 23 pound object being weighed.

Thus we do only need 4 weights of 1, 3, 9, and 27 pounds to weigh objects from 1 to 40 pounds (actually 1 to 53 pounds[2]) if we can put weights in both pans.

References

[1] Dörrie, Heinrich, *100 Great Problems of Elementary Mathematics: Their History And Solution*, Translated By David Antin, Dover Publications. Inc., 1965. (This Dover edition, first published in 1965, is a new translation of the unabridged text of the fifth edition of the work published by the Physica-Verlag, Würzburg, Germany, in 1958 under the title *Triumph der Mathematik: Hundert berühmte Probleme aus* zwei *Jahrtausenden mathematischer Kultur.*)

[2] Dudeney, Henry Ernest, *Amusements in Mathematics*, Thomas Nelson & Sons Ltd, 1917, reprint Dover Publications, 1958

[3] Ball, W. W. Rouse, *Mathematical Recreations And Essays*, 7th edition, Macmillan And Co., London, 1917.

[4] Leonardo of Pisa, *Liber Abaci*, 1202, Laurence E. Sigler, tr., Springer, 2002.

[2] Wrong! I got carried away with all the ternary number idea. 40 pounds is the maximum possible with *four* weights of 1, 3, 9, and 27 pounds. $53 = 27 + 2 \cdot 9 + 2 \cdot 3 + 2 \cdot 1 = 1 \cdot 3^3 + 2 \cdot 3^2 + 2 \cdot 3 + 2 \cdot 1 = 1 \cdot 3^3 + (3-1) \cdot 3^2 + (3-1) \cdot 3 + (3-1) \cdot 1 = 2 \cdot 3^3 - 1 = (3-1) \cdot 3^3 - 1 = 1 \cdot 3^4 - 1 \cdot 3^3 - 1$. So either an extra 27 pound weight is needed or an 81 pound weight. Either way, that requires more than four weights. My mistake. (I realized this after reading Leonardo's solution from 1202 AD.)