Putnam Octagon Problem

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Here is a problem from the famous (infamous?) Putnam exam. Needless to say, I did not solve it in 30 minutes—but at least I solved it (after making a blizzard of arithmetic and trigonometric errors).

(https://mindyourdecisions.com/blog/2018/11/08/incredible-problem-from-the-hardest-test-the-area-of-an-octagon/, retrieved 11/12/2018)

Incredible Problem From The Hardest Test – The Area Of An Octagon

Posted November 8, 2018 By Presh Talwalkar.

The Putnam competition is an annual exam for undergraduate college students in the U.S. and Canada. It consists of 3 hours for 6 problems, then another 3 hours for another 6 problems. That is 6 total hours for 12 problems, or an average of 30 min/problem.

Each question is worth 10 points, for a maximum of 120 points. But the average (median) score is usually about 1 point, even though it is mostly taken by students who have prepared specifically for the exam and are specialized in math. As further evidence, the exam is given on the first Saturday of December, meaning it only attracts students who want to take an optional 6 hour test around the time of college final exams.

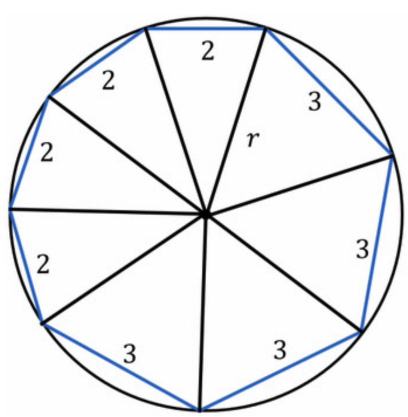
Today's problem is from the 1978 test, problem B1 (the easiest of the second set of problems).

"A convex octagon inscribed in a circle has four consecutive sides of length 3 and four consecutive sides of length 2. Find the area of the octagon."

My Solution

My solution is horribly pedestrian and fraught with numerous chances for arithmetic mistakes to derail it, which happened in spades. As I suspected, there was an elegant, "easy" solution (as demonstrated by Talwalkar below)—*once you thought of it*! Again, this is like a Coffin Problem.[1]

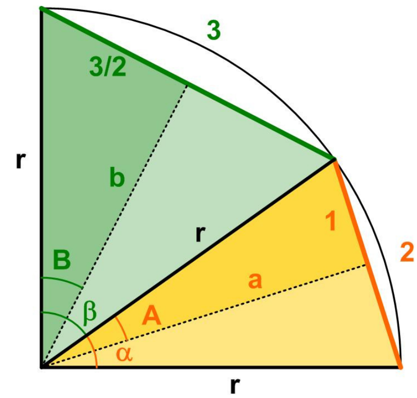
Anyway, here is my pedestrian solution for what it's worth. The following figure shows the statement of the problem:



[1] Khovanova, Tanya, "Coffins" August 2008 (http://www.tanyakhovanova.com/coffins.html).

The first thing to notice is that the octagon is divided into 4 isosceles triangles with base 3 and sides r, and 4 isosceles triangles with base 2 and sides r. If α is the vertex angle of the base 2 triangle and β the vertex angle of the base 3 triangle, then

$$4\alpha + 4\beta = 360° \text{ or } \alpha + \beta = 90°$$



Let $A = \alpha/2$ and $B = \beta/2$. Then $A + B = 45°$. From the figure we have the altitudes of the triangles are given by

$$a = r \cos A$$

$$b = r \cos B.$$

Now $\sin A = 1 / r$ and $\sin B = 3 / 2r$. Therefore,

$$\sin B = (3/2) \sin A = (3/2) \sin (45° - B) = (3/2)[\sin 45° \cos B - \cos 45° \sin B]$$

$$= (3/(2\sqrt{2}))[\cos B - \sin B]$$

And so,

$$(1 + 2\sqrt{2}/3) \sin B = \cos B$$

which means

$$b = r \cos B = (3/(2 \sin B))\,(1 + 2\sqrt{2}/3) \sin B = 3/2 + \sqrt{2}$$

Then the altitude of the base 2 triangle is

$$a = r \cos A = r \cos (45° - B) = r[\cos 45° \cos B - \sin 45° \sin (-B)]$$

$$= (1/\sqrt{2})[r \cos B + r \sin B] = (1/\sqrt{2})[(3/2 + \sqrt{2}) + 3/2] = 1 + (3/2)\sqrt{2}$$

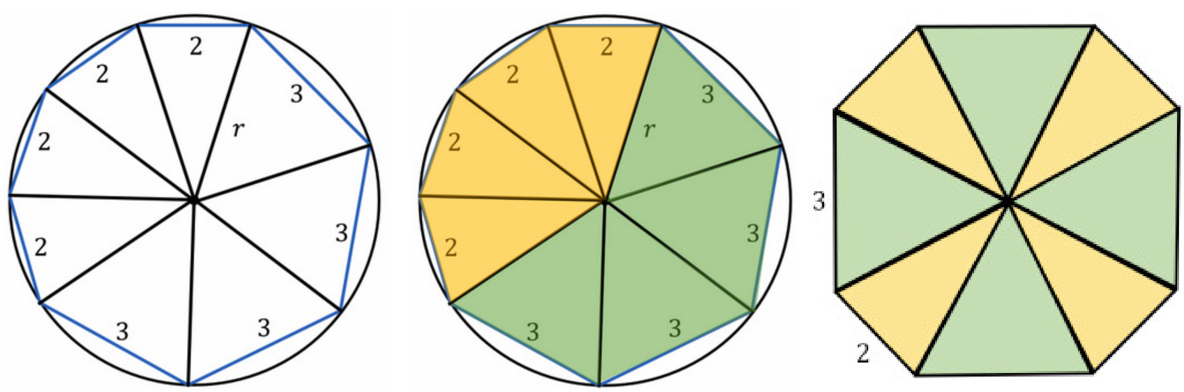
So the area of the octagon is 4 times the sum of the areas of the base 2 and base 3 triangles:

$$\text{Octagon Area} = 4[a\cdot 1 + b\cdot(3/2)] = 4[(1 + (3/2)\sqrt{2}) + (3/2)(3/2 + \sqrt{2})] = 13 + 12\sqrt{2}.$$

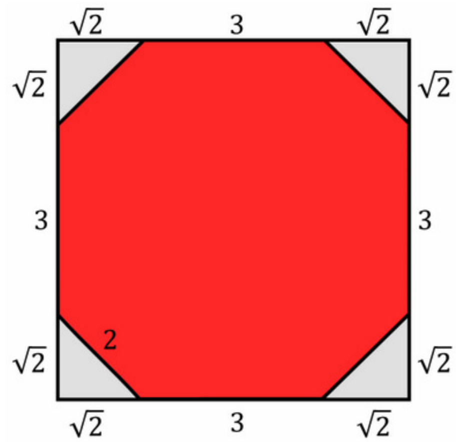
Talwalkar Solution

The problem can be solved easily, if you enjoy eating pizza! I admit I only got as far as sketching the shape. But it was a fun solution to learn!

First let's sketch the octagon, and also draw radial lines from the circle to divide the octagon into 8 "pizza slice" triangles. Let r be the radius of the circle. There are now 4 isosceles triangles with sides r, r, 3, and 4 isosceles triangles with sides r, r, 2. The insight is we can re-arrange these triangles into a new octagon which will have the same area as the original octagon. So let's make an octagon by alternately placing the two triangle shapes with sides of 3 and 2.



How can we solve for this shape's area? We again think outside the box! Draw a square around the octagon. The area of the octagon is then the area of the square minus the 4 corners, which are each isosceles right triangles with a hypotenuse of 2 (and hence legs of $\sqrt{2}$).



The square has a side length of $3 + 2\sqrt{2}$, so we have:

$$\begin{aligned} \text{area(octagon)} &= \text{area(square)} - 4\ \text{area(triangle)} \\ &= (3 + 2\sqrt{2})^2 - 4(0.5(\sqrt{2})(\sqrt{2})) = 17 + 12\sqrt{2} - 4(1) \\ &= 13 + 12\sqrt{2} \end{aligned}$$

And like magic we have calculated the answer!
