Magic Parallelogram

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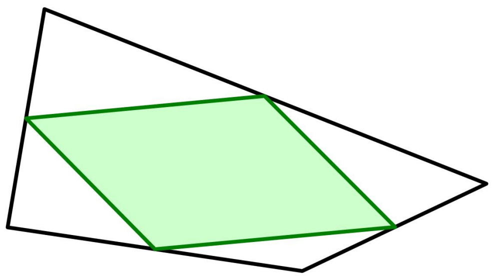

I came across this problem in Alfred Posamentier's book ([1] p.155), but I remember I had seen it a couple of places before and had never thought to solve it. At first, it seems like magic.

In *any* convex quadrilateral (non-intersecting sides) inscribe a second convex quadrilateral with its vertices on the midpoints of the sides of the first quadrilateral. Show that the inscribed quadrilateral must be a parallelogram.

Solution

Once you see the lines added in Figure 1, the approach is obvious if you remember properties of similar triangles from plane geometry. The theorems virtually directly give you the desired parallelism (in particular, Book VI Prop 2, see Appendix below p.3).

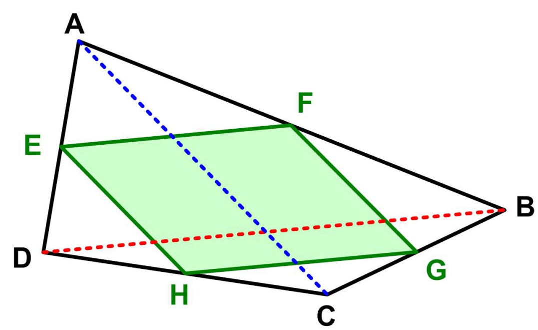

Figure 1 Problem Solution

My problem was that even though I remembered the basic idea, I was always a bit hazy on the details and especially the proofs. I looked up Euclid's *Elements* online and Book VI which dealt with similar triangles ([2] and the Appendix below). Euclid's initial definition of similar triangles said they had equal corresponding angles *and* the corresponding sides were proportional, whereas I had not remembered that both properties were required. But then Props 4 and 5 said that either condition implied the other. Then I realized I could not remember how to prove these things. And when I looked at the proofs, I found them a bit intricate and not all that obvious. Indeed, I always found proofs involving proportions to be a bit opaque.

Rather than just quote the results, I wanted to have a solution that followed from simple arguments. A simple argument by my definition was one involving concepts I could remember after some 50 or 60 years (and that did not include the similar triangle proofs). It turned out that vectors fulfilled this criterion, at least for me. That is, the geometric proof reduced to some arithmetic procedures with vectors, and they are easy to remember. In fact, I already used these ideas in "The Four Travelers Problem" ([3]).

Figure 2 shows the heart of the matter. We extract the triangle ABD from Figure 1 and represent the sides AD and AB by the vectors **u** and **v**, respectively. Then the side DB becomes the vector difference **v** – **u**. The line segment AE in Figure 1 becomes the vector t**u**, where $t = {}^1/_2$, and the line segment AF becomes the vector t**v**. Then the line segment EF is the vector t**v** – t**u**. But

$$t\mathbf{v} - t\mathbf{u} = t(\mathbf{v} - \mathbf{u}) \qquad (*)$$

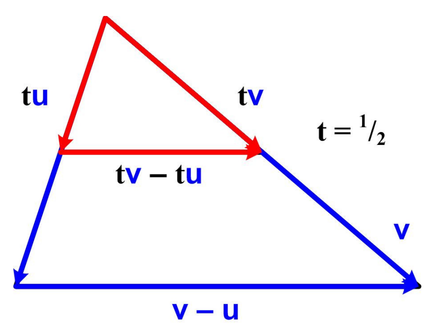

Figure 2 Similar Triangles

(There are actually a bunch of things going on here, which are covered by the definition of a vector space. We have four operations that need to be "compatible", that is, satisfy distributive laws, namely, (1) vector addition, (2) scalar multiplication—multiplication of a vector by a scalar (real number), (3) scalar addition, and (4) scalar multiplication. First, all these operations commute. Then, they satisfy distributive laws:

(1) and (2): $a(\mathbf{v} + \mathbf{w}) = a\mathbf{v} + a\mathbf{w}$

(2) and (3): $(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$

(2) and (4): $(ab)\mathbf{v} = a(b\mathbf{v})$

So in our equation (*) above, we have

$$t\mathbf{v} - t\mathbf{u} = t\mathbf{v} + (-1)(t\mathbf{u}) = t\mathbf{v} + (-1)(t)\mathbf{u} = t(\mathbf{v} + (-1)\mathbf{u}) = t(\mathbf{v} - \mathbf{u})$$

where subtraction is equivalent to adding the additive inverse $-\mathbf{u}$, and $-\mathbf{u} = (-1)\mathbf{u}$.

In other words, the rules which make these vector operations "like" operations with real numbers mean we can "forget about it" just as we do with real numbers—which is further confirmed by using the same notation for technically different operations.)

Now equation (*) gives two things at once:

1. $t\mathbf{v} - t\mathbf{u} = t(\mathbf{v} - \mathbf{u}) \parallel \mathbf{v} - \mathbf{u}$, since any scalar multiple of a vector is parallel to that vector (or if you remember things about the vector cross product $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$ (so $\mathbf{u} \times \mathbf{u} = \mathbf{0}$) and $\mathbf{u} \times \mathbf{v} = \mathbf{0} \Rightarrow \mathbf{u} \parallel \mathbf{v}$, then $(t\mathbf{u}) \times \mathbf{u} = t(\mathbf{u} \times \mathbf{u}) = \mathbf{0}$ proves the statement). So in our case EF || DB.
2. The length of $t\mathbf{v} - t\mathbf{u}$ (EF in our case) is t = ½ the length of $\mathbf{v} - \mathbf{u}$ (DB in our case).

We can use this same argument on the triangle CBD with line segment GH connecting the midpoints to yield GH || DB and GH = ½ of DB. Therefore, GH || DB || EF and GH = ½ DB = EF. Repeat this argument for the triangles ABC and CDA to yield FG || HE and FG = HE.

Hence, EFGH is a parallelogram.[1]

Appendix

Euclid's *Elements* Book VI ([2])

Definitions

Definition 1.

Similar rectilinear figures are such as have their angles severally equal and the sides about the equal angles proportional.[2]

…

Definition 4.

The height of any figure is the perpendicular drawn from the vertex to the base.

[1] JOS: Actually, it is sufficient to show the pairs of midpoint-connecting lines are parallel, since pairs of intersecting parallel lines cut off corresponding line segments of equal length, and thus form a parallelogram. The proof is simple and LTR.

[2] JOS: Props 4 and 5 show either property alone is sufficient to define similar triangles.

Propositions

Proposition 1.

Triangles and parallelograms which are under the same height are to one another as their bases.

Proposition 2.

If a straight line is drawn parallel to one of the sides of a triangle, then it cuts the sides of the triangle proportionally; and, if the sides of the triangle are cut proportionally, then the line joining the points of section is parallel to the remaining side of the triangle.

Proposition 3.

If an angle of a triangle is bisected by a straight line cutting the base, then the segments of the base have the same ratio as the remaining sides of the triangle; and, if segments of the base have the same ratio as the remaining sides of the triangle, then the straight line joining the vertex to the point of section bisects the angle of the triangle.

Proposition 4.

In equiangular triangles the sides about the equal angles are proportional where the corresponding sides are opposite the equal angles.

Proposition 5.

If two triangles have their sides proportional, then the triangles are equiangular with the equal angles opposite the corresponding sides.

Proposition 6.

If two triangles have one angle equal to one angle and the sides about the equal angles proportional, then the triangles are equiangular and have those angles equal opposite the corresponding sides.

References

[1] Posamentier, Alfred S., *Math Charmers: Tantalizing Tidbits for the Mind*, Prometheus Books, New York, 2003

[2] Joyce, David E., "Book VI," *Euclid's Elements,* 1998 (https://mathcs.clarku.edu/~DJoyce/java/elements/bookVI/bookVI.html)

[3] Stevenson, James, "The Four Travelers Problem." *Meditations on Mathematics*, 2016.