Infinite Product Problem

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This is a challenging problem from *Mathematical Quickies* (1967) ([1] p.37).

129. Evaluate the infinite product:

$$3^{1/3}\cdot 9^{1/9}\cdot 27^{1/27}\cdot\ldots\cdot(3^n)^{1/3^n}\cdot\ldots$$

My Solution

Mathematical Quickies provides a solution that does not use calculus (see below), but I like the calculus approach. It uses techniques that are standard for infinite products and the related infinite series.

We first convert the infinite product P to an infinite series by taking the natural log of P, ln P:

$$\ln\prod_{n=1}^{\infty}(3^n)^{1/3^n} = \sum_{n=1}^{\infty}\ln(3^n)^{1/3^n} = \sum_{n=1}^{\infty}\frac{n\ln 3}{3^n} = \ln 3\sum_{n=1}^{\infty}\frac{n}{3^n}$$

The last infinite series is of the form

$$f(x) = \sum_{n=1}^{\infty} nx^n = x + 2x^2 + 3x^3 + \ldots$$

where $x = 1/3$. But this is reminiscent of a derivative. That is, if

$$F(x) = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n,$$

(the geometric series), then

$$F'(x) = 1 + 2x + 3x^2 + 4x^3 + \ldots = \sum_{n=1}^{\infty} nx^{n-1}$$

and

$$xF'(x) = x + 2x^2 + 3x^3 + \ldots = \sum_{n=1}^{\infty} nx^n = f(x).$$

But the geometric series has a sum $F(x) = 1/(1-x)$ for $|x| < 1$. Therefore $F'(x) = 1/(1-x)^2$ and $xF'(x) = x/(1-x)^2 = f(x)$. Setting $x = 1/3$ yields $f(1/3) = (1/3)/(2/3)^2 = 3/4$. The original series was

$$\ln 3\, f(1/3) = (3/4)\ln 3 = \ln (3)^{3/4}$$

Raising this expression to the power of e yields the original infinite product P, namely, $P = 3^{3/4}$.

These steps look complicated, but they are typical of problems involving power series, since they can be differentiated or integrated term by term. The geometric series is the workhorse power series (because its sum is known) and often the approach to solving problems is to see how through multiplications, differentiation, and/or integration of the geometric series the target series can be achieved.

Mathematical Quickies Solution

Let

$$N = 3^{1/3} \cdot 9^{1/9} \cdot 27^{1/27} \cdot \ldots \cdot (3^n)^{1/3^n} \cdot \ldots = 3^{1/3+2/9+3/27+\ldots+n/3^n+\ldots} = 3^M$$ [1]

Then

$$\frac{1}{3}M = \frac{1}{3^2} + \frac{2}{3^3} + \frac{3}{3^4} + \ldots + \frac{n-1}{3^n} + \ldots$$ [2]

So

$$\left(1 - \frac{1}{3}\right)M = \left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \ldots + \left(\frac{1}{3}\right)^n + \ldots$$ [3]

$$= \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{1}{2}.$$ [4]

Hence, $N = 3^{3/4}$.

References

[1] Trigg, Charles W., *Mathematical Quickies: 270 Stimulating Problems with Solutions*, McGraw-Hill Publ, New York, 1967, corrected ed, Dover Publ., Mineola, New York, 1985.

[1] JOS: This is a way of avoiding mentioning logarithms explicitly.

[2] JOS: The reason for multiplying by 1/3 will become evident in the next step.

[3] JOS: This is a standard trick for finding the sum of a geometric series. Applied to this different series, it just happens to convert it into a geometric series, which then has a sum. I think these steps come across as a trick without sufficient motivation. The steps I used at least follow a motivated scheme that works in many cases.

[4] JOS: $M = (^1/_3)/(1 - {}^1/_3)^2 = (^1/_3)/(^2/_3)^2 = {}^3/_4$, which is the step I got with $f(1/3)$.